Elasto-plastic analysis of thin shells under thermal transients

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1 ABSTRACT

The objective of this paper is to use a thermal-elasto-plastic algorithm developed by the authors in order to find the temperatures and thermal stresses that are produced by time-dependent thermal loads, such as the ambient temperatures and prescribed surface heat fluxes. These loads must be given as functions of time. With the thermal conductivities and surface film coefficients given for heat conduction, and the elastic and plastic material properties for stress analysis, the outcome is the temperature, displacement, and stress distributions that are produced by the thermal transients. We apply the analysis to thin shells; that is, objects that have one dimension, wall thickness, smaller than any other length. Even though the ideas given here apply to shells of general shape, we limit the analysis in this paper to cylindrical shells and consider axisymmetric thermal loads only.

2 INTRODUCTION

For the prediction of stresses and displacements of a shell subjected to mechanical loads, it is a common practice to use what is called "shell theory". The basic idea in shell theory is that, owing to the small thickness of the wall, the integrated governing equations across the wall are satisfied. The advantage of this approach is that the number of independent variables is reduced from three to two. The objective of the present paper is to show how the same advantage can be achieved for heat conduction in a thin shell.

3 BOUNDARY VALUE PROBLEM OF HEAT CONDUCTION

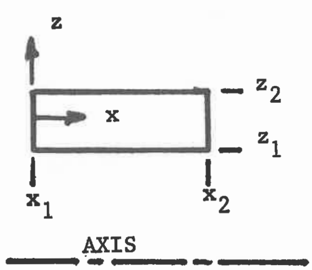

Figure 1

We consider a cylindrical shell and a coordinate system as shown in Figure 1. The y axis goes into the plane of the paper. Since we will consider axisymmetric problems only, there is no temperature gradient, and thus no heat flow, in the y direction. The direction along x will be called "axial", because it is along the axis of symmetry of the shell, and that along z will be referred to as the "across-the-wall" direction.

A boundary value problem of heat conduction is defined with respect to a region that is bounded by the bounding surfaces of the shell, at $z=z_1$ and $z=z_2$, and the shell edges, at $x=x_1$ and $x=x_2$. The objective is to find the temperature field within this region, subject to some boundary conditions at the bounding surfaces and edges.

3.1 The governing equations within the region

The Fourier Law of heat conduction along x and z is given by

(1) $$q_x=-k_x T_{,x}$$ (2) $$q_z=-k_z T_{,z}$$

where q_x and q_z are the axial and across-the-wall the heat fluxes (heat per area per time), k_x and k_z are the thermal conductivities of the material, T is the temperature, and a comma denotes differentiation with respect to the subscript that follows the comma.

The balance of the rate at which heat flows into and out of, and is retained within, an element of volume dxdydz, is given by

(3) $$q_{x,x}+q_{z,z}+rcT_{,t}=0$$

where r denotes the mass density, c is the specific heat of the material, and t is time.

3.2 Boundary conditions at the surfaces

Since the governing equations in each of the directions, x and z, are of second order, and since the boundary variables at the surfaces x=constant are q_x and T, and those at the surfaces z=constant are q_z and T, then it follows that the most general boundary conditions are:

(4) $$q_x=-h_{x2}(T_{x2}-T)+q_{x2} \quad \text{at } x=x_2$$

and $$q_x=-h_{x1}(T-T_{x1})+q_{x1} \quad \text{at } x=x_1$$

and

(5) $$q_z=-h_{z2}(T_{z2}-T)+q_{z2} \quad \text{at } z=z_2$$

and $$q_z=-h_{z1}(T-T_{z1})+q_{z1} \quad \text{at } z=z_1$$

where the h's, T's, and q's, with double subscripts, can be arbitrary numbers. In heat conduction, the h's are called the surface film coefficients (also surface conductances or coefficients of surface heat transfer), the T's are the ambient temperatures near the surfaces that are designated by their subscripts, and the q's are heat fluxes that are applied directly to the surfaces.

In an actual problem, the q's could come from some radiation coming from the outside of the material, or generated within the shell material near the surfaces, as in induction heating. The film coefficients could come from a degradation of the surface material, as for example from unwanted deposits of oxides, or from wanted deposits, such as cladding, or from the boundary layer of a viscous fluid that is flowing past the surface.

4 RESULTANT FLUX AND AVERAGE TEMPERATURE

Owing to the special geometry (small thickness), it is useful to introduce integrated variables across the wall. This is commonly done in shell theory, which leads there to stress resultants, stress couples (moments), and an average normal deflection. We will do the same in heat conduction. We will define the resultant flux, Q_x, by

$$(6) \qquad Q_x = \int_1^2 q_x dz$$

which is the axial heat flow rate per unit length (not area) of the edge defined by x=constant that is perpendicular to the plane of the paper in Figure 1. For example, in a cylindrical shell of a mean radius R, the total amount of heat per second that passes through an annular edge, defined by x=constant, is

$$(7) \qquad \text{HEAT FLOW RATE}=6.2832RQ_x \text{ BTU/sec}$$

Similarly, we introduce an integrated measure of temperature. It is its average, u, across the wall, given by

$$(8) \qquad u=\frac{1}{h} \int_1^2 Tdz$$

where h denotes the thickness of the wall ($h=z_2-z_1$).

5 FIN MODEL

Standard texts on heat conduction (for example, Eckert and Drake, 1959, pages 43-58) develop models for the conduction of heat in finned surfaces, usually used for the cooling of an object. Such a "fin model" can be obtained by the integration of equations (1) and (3) with respect to z, which, for the steady-state case, become

$$(9) \qquad Q_x=-hk_x u_{,x} \quad \text{and} \quad Q_{x,x}+(h_{z2}+h_{z1})u-(h_{z2}T_{z2}+h_{z1}T_{z1})=0$$

Equation (2) is not used, which means that heat flow across the wall is not modelled. Note that, if the h's are the same, then only the sum of the ambient temperatures matters but not their difference. The difference is what is missing in the fin model.

This is a second-order system in two unknowns, Q_x and u, that is solved subject to the boundary conditions given by equations (4) at the edges of the shell. The temperature is modelled by its average value, u, and is assumed uniform across the wall. The heat flow (usually heat loss) across the bounding surfaces is then calculated from equations (5) by substituting u for T.

The fin model works well if the main objective of the analysis is the estimate of the total heat loss away from the fin, and it is not important to predict the transverse flux, q_z, inside the wall. It is not adequate if temperature gradients across the wall are of importance, which is the case in a subsequent stress analysis.

6 A NEW ALGORITHM OF HEAT CONDUCTION IN SHELLS

6.1 Assumed form of temperature field

The basic idea of the new algorithm is the expression of the temperature field in the form

(10) $$T(x,z)=f(z)U(x)+V(z,x)$$

which allows for the separation of the two-dimensional problem into two one-dimensional problems: an axial problem, represented by U(x), and an across-the-wall problem, represented by V(z,x). f(z) can be viewed as a "form function" that is needed to model heat flux away from the bounding surfaces in the axial problem.

6.2 Across-the-wall problem

Substitution of T from equation (10) into (3), and using (2), gives

(11) $$q_{x,x}-k_z f_{,zz}U-(k_z V_{,z})_{,z}+rcfU_{,t}+rcV_{,t}=0$$

The across-the-wall problem is defined by setting the sum of the third an fifth terms equal to zero. Thus, V is defined as the solution of

(12) $$q'_{z,z}+rcV_{,t}=0 \qquad \text{where} \qquad q'_z=-k_z V_{,z}$$

subject to the boundary conditions

(13) $$q'_z=-h_{z2}(T_{z2}-V)+q_{z2} \qquad \text{at } z=z_2$$

$$\text{and} \qquad q'_z=-h_{z1}(V-T_{z1})+q_{z1} \qquad \text{at } z=z_1$$

Even though V is said to be a function of z and x, its x-dependence occurs only through the nonhomogeneous terms: the ambient temperatures, T_{z1} and T_{z2}, and the applied surface fluxes, q_{z1} and q_{z2}. If these terms vary in the axial direction, then V can be determined, as a function of z, at specified points along x. If these terms are constants, then V must be determined only at one value of x.

This leaves equation (11) in the form

$$(14) \qquad q_{x,x}-k_z f_{,zz}U+rcfU_{,t}=0$$

which will be solved by the axial problem.

6.3 Preparation for the Axial problem

Integration of equation (14) with respect to z yields

$$(15) \qquad Q_{x,x}+h'_e U+hrcU_{,t}=0 \qquad \text{where } h'_e=-[(k_z f_{,z})]_{z=z2}+[(k_z f_{,z})]_{z=z1}$$

where we have assumed that

$$(16) \qquad \frac{1}{h}\int_1^2 f dz=1$$

We will be able to determine more precisely the role of the form function f, if we integrated also equation (3) with respect to z directly, without specifying the form of q_z. The result is given by

$$(17) \qquad Q_{x,x}+[q_z]_{z=z2}-[q_z]_{z=z1}+hrcu_{,t}=0$$

where the surface heat flux terms are given by equations (5).

Since equations (15) and (17) both define axial heat flow, we must show that they can be made identical. For that purpose, we must first find the relationship between the average temperature and the new variables introduced in equation (10). Substitution of T from (10) into (8) and the use of (16) gives the relation

$$(18) \qquad u=U+T_e \qquad \text{where} \qquad (19) \qquad T_e=\frac{1}{h}\int_1^2 V dz$$

Substitution of u from (18), the surface fluxes from (5), and T from (10), into equation (17) yields

$$(20) \qquad Q_{x,x}+h_e U+hrcU_{,t}+[q'_z]_{z=z2}-[q'_z]_{z=z1}+hrcT_{e,t}=0$$

where $\quad h_e=h_{z2}[f]_{z=z2}+h_{z1}[f]_{z=z1}$

and the q' terms are defined by (13).

But the last three terms of equation (20) are zero from the across-the-wall problem. They represent the balance of the heat flow rate of a shell element that is bounded by the bounding surfaces and two neighboring planes defined by x=constant. Equation (20) then becomes

$$(21) \qquad Q_{x,x}+h_e U+hrcU_{,t}=0$$

which is the same as (15) if we can identify $h_e=h'_e$. This identity can be enforced by the determination of the form function f.

6.4 Definition of the axial problem

We need also the relationship for the axial heat flux and average temperature that we already used in the fin model, as given by the first of equations (9). The governing equations of the axial problem consist of the first of (9), (18), and (21); they contain three unknowns: Q_x, u, and U. Our procedure requires that the across-the-wall problem be solved first. Then V(z,x) is known, and T_e can be determined before the axial problem is started. Therefore, as far as the axial problem is concerned, T_e is regarded as known.

Since the three equations contain the derivatives of two variables with respect to x, it clear that only two boundary conditions, containing the linear combinations of two boundary variables, must be prescribed on the edges defined by x=constant. The equations contain a derivative of Q_x; therefore, Q_x must be one of the boundary variables. However, we could make either u or U the second boundary variable. The obvious choice is the average temperature, u, because, in a description of a physical heat conduction problem, this quantity can be easily computed at an edge x=constant. This is not the case with U.

Therefore, we eliminate U from equations (18) and (21), which then, together with the first of (9), forms the governing system of equations for axial heat flow:

(22) $$Q_x=-hk_x u_{,x} \quad \text{and} \quad Q_{x,x}+h_e(u-T_e)+hrc(u-T_e)_{,t}=0$$

where h is the thickness of the wall. The boundary conditions are obtained by the integration of equations (4) with respect to z across the wall. The result is

(23) $$Q_x=h\{-h_{x2}(T^*_{x2}-u)+q^*_{x2}\} \quad \text{at } x=x_2$$

$$\text{and} \quad Q_x=h\{-h_{x1}(u-T^*_{x1})+q^*_{x1}\} \quad \text{at } x=x_1$$

where the T*'s and the q*'s denote the average values of the ambient temperatures and the applied surface fluxes across the wall at the edges, like the average temperature, which is defined by equation (8).

7 DETERMINATION OF FORM FUNCTION

The form function f(z) must satisfy the condition that its average across the wall be unity and that $h_e=h'_e$. We choose to determine it as the solution of a boundary value problem that is governed by

(24) $$k_z f_{,zz}+k_x b=0$$

and the following boundary conditions

(25) $$k_z f_{,z}=-h_{z2}f \quad \text{at } z=z_2$$

and $\quad k_z f_{,z} = h_{z1} f \quad$ at $z=z_1$

which arise from the identification of the h_e's in (15) and (20). The constant b in (24) is chosen in such a way that (16) is satisfied.

8 TRANSIENT HEAT CONDUCTION

8.1 Governing equations

The time derivatives that occur in the second of equations (22) are replaced by finite differences in time. We choose a weighted residual scheme that is described by Zienkiewicz (1977) on pages 570-580.

We define a new variable G by

(26) $$G=-\{Q_{x,x}+h_e(u-T_e)\}$$

so that the second of equations (22) can be written as

(27) $$hrc(u-T_e)_{,t}=G$$

We now consider a time interval, Δt, beginning at $t=t_0$ and ending at $t=t_1$, so that $\Delta t=t_1-t_0$, and identify variables at t_0 by a subscript 0 and those at the end of the interval by no subscript. The weighted residual scheme replaces the derivatives with respect to time with a weighted average of their values. Thus the slope of $u-T_e$ within the interval is equated to a weighted average of G at t_0 and t_1, as in

(28) $$hrc[u-T_e-(u_0-T_{e0})]/\Delta t=wG+(1-w)G_0$$

where w is the weighting parameter. The problem is solved for the variables at the end of the interval (with no subcripts) in terms of those at the beginning of the interval (with subscript 0).

Thus, replacing G from (26) into equation (28) yields a new governing equation that replaces the second of (22) in the form

(29) $$w\{Q_{x,x}+h_e(u-T_e)\}+hrc[u-T_e-(u_0-T_{e0})]/\Delta t-(1-w)G_0=0$$

The pair of equations, the first of (22) and (29), represent the system of governing equations of the transient problem of axial heat flow. We solve for the derivatives and write them in the form

(30) $$Q_{x,x}=-H_e(u-T_e)+rc(u_0-T_{e0})/(w\Delta t)+(1-w)G_0/w \qquad u_{,x}=-Q_x/hk_x$$

where $H_e=h_e+hrc/(w\Delta t)$. As before, the unknowns are Q_x and u, while T_e and all the variables with a subscript 0 are regarded as known. The boundary conditions on Q_x and u remain those given by equations (23). For the starting point in time, say at t=0, we determine the solution for u_0, T_{e0}, and G_0 by solving the steady state problem, using whatever input variables (ambient temperatures, edge conditions) are acting at t=0. The transient problem begins with the determination of the solution at the end of the first time interval.

Exactly the same approach is followed in the transient problem for heat conduction across the wall, which is defined by equations (12). Following the same steps as those given in this section, the governing equations of the across-the-wall problem are written in the form

$$(31) \qquad q'_{z,z}=-rc(V-V_0)/(w\Delta t)-(1-w)[q'_{z,z}]_0/w \qquad V_{,z}=-q'_z/k_z$$

The boundary conditions remain those given by equations (13).

8.2 Time step

The weighted residual scheme is classified as an "implicit" scheme (Zienkiewicz, 1977, page 572), meaning that at every time step a new boundary value problem must be solved, which is of the same type as that of a steady-state problem. Such implicit schemes are stable for large time steps, but may lead to unacceptably large spurious oscillations if the time step is too small. According to Zienkiewicz (1977), page 579, the value of w=.6667 is said to be the best choice because then "... almost all oscillation and errors are avoided."

Our experience has shown that even with w=.6667 there must be a minimum time step below which the results are questionable. Such a minimum step can be based on setting a minimum value of the Fourier modulus (Eckert and Drake, 1959, page 76), defined by

$$(32) \qquad FMOD=k\Delta t/(rcL^2)$$

where L is some characteristic length of the problem, and k could be either the axial or across-the-wall conductivity. Our experimentation with time steps of various sizes has shown that the minimum time step should be about that when FMOD=0.06, which gives the time step of

$$(33) \qquad \Delta t_{min}=0.06rcL^2/k$$

A reasonable value for L is the thickness of the wall.

9 SOLUTION OF THE BOUNDARY VALUE PROBLEM

The procedure of calculation consists of solving first the across-the-wall problem, as defined by equations (31), for one time point, storing the necessary variables at specified stations along x, and then solving the axial problem, as defined by equations (30).

Both systems of equations are of the form that can be solved by a direct numerical integration method that has been advocated for many years by Kalnins (1964). This method can be used for the across-the-wall problem, but, for the axial problem, it fails if SK, which is the ratio of the average axial conductivity over an effective film coefficient, is too small in comparison with the length of the axial dimension. SK represents the length of a boundary layer at the edge of the shell. For boundary value problems with very short boundary layers, the field method developed by Cohen (1974) works much better. In the extreme case, if SK is very small, then the axial boundary value problem need not be solved at all, because the axial heat flow can then be set equal to zero in the whole region and the average temperature equated to the average of the ambient temperatures near the bounding surfaces. What happens physically is that the heat loss across the bounding surfaces overwhelms the slow axial heat flow, and all the action takes place near the starting edge.

10 VERIFICATION OF THE ALGORITHM

The heat conduction algorithm and the solution procedure has been incorporated in a computer program called KSHELH. The following solutions were calculated with KSHELH. We have chosen two cases that can be compared with an exact analytical solution. The geometry is like that in Figure 1, with the axial length L=1 and thickness h=1. The material properties are: $k_x=k_z=rc=1$.

10.1 Transient heat conduction along axis

The film coefficients are prescribed very small, $h_{z1}=h_{z2}=.0001$, to insulate the bounding surfaces. Temperature is prescribed on the edges and modelled as a ramp function of time. The values of the prescribed temperatures at each time step are given in TABLE 1, in the row marked TEMP. The transient solution is calculated at seven time points, up to t=0.42, with a time step of 0.06. TABLE 1 gives the values of the temperature at the middle of the element, at x=0.5, in the row marked KSHELH-1.

10.2 Transient heat conduction across the wall

The film coefficients are now very large, $h_{z1}=h_{z2}=10{,}000$, to make the surface temperatures the same as the ambients, which are again modelled as a ramp function of time and given in TABLE 1. The fluxes at the edges are prescribed zero (insulated). TABLE 1 gives the temperatures at the middle of the element, at z=0, in the row marked KSHELH-2.

TABLE 1. Verification of transient heat conduction.

TIME	0.0	.06	.12	.18	.24	.30	.36	.42
TEMP	0.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
KSHELH-1	0.0	16.28	47.10	70.78	83.42	90.59	94.68	96.97
KSHELH-2	0.0	17.66	47.68	70.84	83.38	90.51	94.60	96.93
EXACT	0.0	10.55	46.9	70.6	83.7	91.0	95.0	97.2

10.3 Comparison

Both problems are the same if the cross section is turned by 90 degrees. Therefore, the same exact solution applies. The overshoot of the KSHELH solution comes from the sharp change from zero, to the ramp, to the constant temperature at time t=0.06, all in one step.

11 APPLICATION TO ELASTO-PLASTIC PROBLEMS

After the temperature field within the shell is determined at a number of points in time, this result is transfered to an elasto-plastic stress analysis algorithm for the calculation of the displacements and thermal stresses. We use the algorithm of the KSHEL computer program, which is described in a paper by Kalnins and Updike (1985). This algorithm has now been expanded further to allow for temperature-dependent elasto-plastic material properties. The transient time-point input is ideally suited for the incremental plasticity analysis, because one time point can be equated with one step in the plasticity problem.

To illustrate the use of the algorithm , we have chosen the induction heating stress improvement process (IHSI) of piping, which is described in two EPRI reports: Rybicki et al (1982) and Offer (1983). A report on its recent progress is given by Cofie and Sheffield (1985).

12 EXAMPLE: INDUCTION HEATING STRESS IMPROVEMENT

12.1 IHSI in brief

The IHSI process consists of the heating by electromagnetic induction of the material near the outer surface (OD) of a pipe, while the inner surface (ID) is cooled by water. After some time of operation, the heater is turned off, and the pipe is allowed to cool to a uniform temperature. The result of the process is that the pipe retains residual stresses that are compressive on the ID, which makes the ID more resistant to intergranular stress corrosion cracking.

The IHSI process is well suited for our thermal-elasto-plastic algorithm. A prescribed time history of heat flux is applied to the outer surface of a cylindrical shell, and from this input KSHELH produces the material temperatures at a number of time points. Using the known temperatures as input, KSHEL then produces the corresponding stress states at the same time points.

As shown by Rybicki et al (1982), induction heating generates heat near the OD with a density (BTU per volume per time) that falls off when going from the OD toward the ID. In order to simplify our example, we have assumed that all the heat is applied at the OD surface itself. Also, we have assumed that the pipe is in a plane strain state, which corresponds physically to a pipe and heater of infinite length. We assume a pipe with a mean radius of 10, wall thickness of 1, and the length of 2, all in inches.

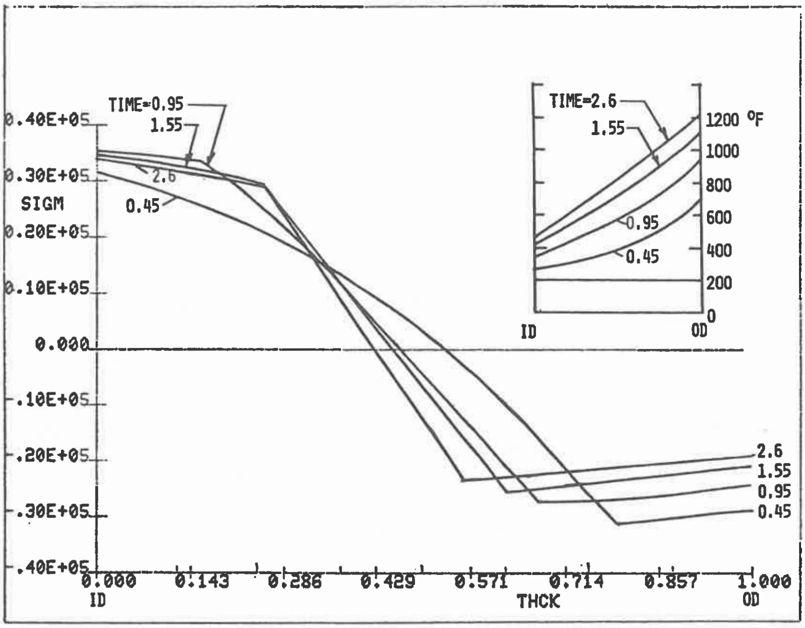

Figure 2: Heating phase. Distribution of stress, in psi, across wall, at times shown. Insets: temperature distributions at times shown.

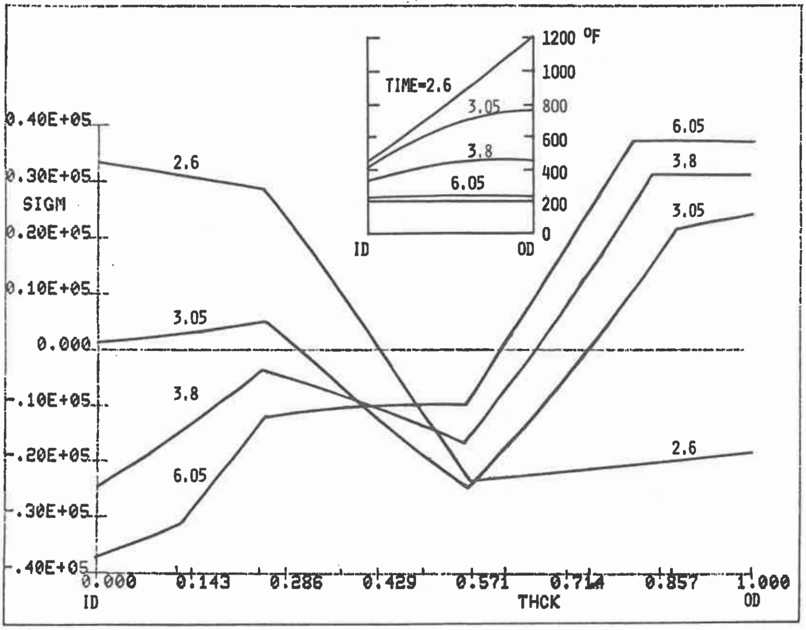

Figure 3: Cooling phase. Distribution of stress, in psi, across wall, at times shown. Insets: temperature distributions at times shown.

12.2 Description of problem

Heat conduction: Thermal conductivity, k=.021 BTU/in./min/deg.F.; density times specific heat, rc=.031 BTU/cubic in./deg.F.; zero heat flux across edges; film coeff. at ID, h_{z1}=0.0579 BTU/sq.in./min./deg.F; at OD, h_{z2}=0.00116; ambient temperatures, $T_{z1}=T_{z2}$=200 deg.F.; heat generation by electromagnetic induction within the wall, assumed to be applied directly to the OD surface, maximum of q_{z2}=-18.06 BTU/sq.in./min. The heating phase consists of applying the surface flux from zero to the maximum in 0.2 minutes and holding the maximum until time t=2.6 minutes, at which time the heater is turned off. The applied flux is assumed to reach zero at t=2.8 minutes and then remain zero.

Stress analysis: At both edges, shear and axial stress resultants, and rotation are specified zero; modulus of elasticity, E=27,000 ksi; Poisson's Ratio, ν=0.28; coefficient of thermal expansion, $\alpha=6.3\text{x}10^{-6}$ per deg.F.; plastic strain-stress points are: at T=200 deg.F.: (.0,38.0 ksi) and (.005,40.0 ksi); at T=1000 deg.F.: (.0,21.0 ksi) and (.005,30.0 ksi). Linear interpolation is used between these points.

12.3 Results

Figure 2 shows the heating phase, from t=0 to t=2.6 (all times are in minutes). The transient problem begins at t=0, from a uniformly heated state at 200 deg.F. At t=.45, the ID is elastic but the OD is already plastic (the sharp knees in the Figures designate the elastic-plastic boundaries). At t=0.95, the ID is also plastic. At t=2.6, the heater is switched off, and the scene moves to Figure 3. We note that the temperature (see inset) and the stress change rapidly. On the ID, within 0.45 minutes, the stress falls from 33 ksi in tension to zero, and, after another 0.75 minutes, it is already -25 ksi in compression. Similar rapid changes take place on the OD. Within 1.2 minutes, the temperature drops from 1200 deg.F. to 475 deg.F., while the stress goes from -18 ksi to 31 ksi. After cooldown, Figure 3 shows these residual stresses: on the ID, -37 ksi in compression and on the OD, 37 ksi in tension. The final residual stress distribution, at t=6.05, is similar to that of Figure III-11 on page 3-20 of Rybicki et al (1982).

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