Automorphism Groups of Balanced Incomplete Block Designs and Their Use in Statistical Model Construction and Analysis

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CATHERINE T. BURTON. Automorphism Groups of Balanced Incomplete Block Designs and Their Use in Statistical Model Construction and Analysis. (Under the direction of I. M. CHAKRAVARTI.)

The analysis of variance of a BIB design using either of the conventional linear models $E(\underline{Y}) = \mu \underline{j}_n + D'\underline{\tau} + L'\underline{\beta}$, $Var(\underline{Y}) = \sigma^2 I$ or $E(\underline{Y}) = \mu \underline{j}_n + D'\underline{\tau}$, $Var(\underline{Y}) = \sigma^2 I + \sigma_1^2 L'L$ does not take into account certain structural properties of the design, such as the automorphism group. We assume here that the covariance matrix V of $\underline{Y}$ is invariant under the action of the group of permutation matrices which permute the flags, the treatment-block pairs, under the action of the group of automorphisms G. Thus $V = \sum_{i=1}^{t} c_i V_i$, where t is the dimension and V_i, $i = 1, 2, \ldots, t$, is a linear basis of the commutant algebra corresponding to the permutation representation of G.

For the desarguesian finite projective plane PG(2, 2) and the desarguesian finite affine plane EG(2, 3), the decompositions into direct sums of irreducible representations of the permutation representations of the full collineation groups PC(2, 2) and EC(2, 3) have been determined. The dimensions and the components of the covariance matrices V of designs derived from such planes are thus determined.

Using group theoretic and combinational results, if the flags are the incident point-line pairs, the dimensions of the commutant algebras for the desarguesian finite projective plane PG(2, s) and the desarguesian finite affine plane EG(2, s) are found to be six and seven, respectively, for every prime p and every integer r, $s = p^r$. If the flags are taken to be the incident point-hyperplane pairs, the dimensions of the commutant algebras for the finite projective geometry

PG(k, s), $k \geq 3$ and the finite affine geometry EG(k, s), $k \geq 3$ are seven and eight, respectively. A linear basis for each case has also been derived.

Estimation is considered when the covariance matrices belongs to one of four relationship algebras R_6, R_7, P_7 or P_8, algebras having basis matrices which reflect relationships between the flags. For the fixed effects model, the best linear unbiased estimator of $E(\underline{Y})$ is equal to the simple least squares estimator of $E(\underline{Y})$; variance component estimation is also examined. For the random blocks model, the estimable components of variance are determined along with the difference between the b.l.u.e. and the l.s.e. of $E(\underline{Y})$.

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I. INTRODUCTION

An *incidence structure* (see, for instance, Dembowski (1968)) is defined to be a triple $S = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ where $\mathcal{P}$, $\mathcal{B}$ and $\mathcal{I}$ are sets such that $\mathcal{P} \cap \mathcal{B} = \emptyset$ and $\mathcal{I} \subseteq \mathcal{P} \times \mathcal{B}$. The elements of $\mathcal{P}$, $\mathcal{B}$ and $\mathcal{I}$ are called *points*, *blocks* and *flags* respectively. A flag will be denoted by an ordered pair (p, B) which indicates that the point p is incident with the block B. We shall also only consider finite incidence structures, i.e., the case when $\mathcal{P}$, $\mathcal{B}$ and $\mathcal{I}$ are finite sets.

A *tactical configuration* is an incidence structure with an equal number of points on each block and an equal number of blocks through every point. Such a tactical configuration is called a *balanced incomplete block design* (BIB design) if every pair of points is incident with an equal number of blocks, say λ (>0). This results in a design in which $v = |\mathcal{P}|$ treatments are arranged in $b = |\mathcal{B}|$ blocks, such that each block contains k different treatments, each treatment occurs in r blocks and every pair of different treatments occurs together in λ blocks (see, for example, Hall (1967)). The results of an experiment based on such a BIB design with parameters v, b, r, k, λ may be written as the vector of observations $\underline{Y} = (Y_1, Y_2, \ldots, Y_n)'$ where $n = bk = vr$ is the number of experimental units.

When the treatments and blocks are fixed, the linear model $E(\underline{Y}) = \mu \underline{j}_n + D'\underline{\tau} + L'\underline{\beta}$, $\text{var}(\underline{Y}) = \sigma^2 I_n$ is commonly used by statisticians in the analysis of variance. Here $\underline{\tau}$ designates the vector of treatment

parameters, $\underline{\beta}$ the vector of block parameters and μ a constant. $\underline{j}_n$ is a vector of n ones. $D = ((d_{iu}))$ is a $v \times n$ matrix such that $d_{iu} = 1$ if the u^{th} experimental unit (flag) receives (contains) treatment i (point i); otherwise, $d_{iu} = 0$. $L = ((l_{ju}))$ is a $b \times n$ matrix such that $l_{ju} = 1$ if the u^{th} experimental unit (flag) is found in block j; otherwise, $l_{ju} = 0$. In the case of random block effects, the linear model $E(Y) = \mu \underline{j}_n + D'\underline{\tau}$, var $(\underline{Y}) = \sigma^2 + \sigma_1^2 L'L$ is commonly used. Such linear models, however, do not discriminate between two BIB designs with the same numerical values for the parameters v, b, r, k, λ but with different structural properties. Combinational mathematicians have studied the differences in structure of these BIB designs by examining the *groups of automorphisms* (groups of symmetries) of such designs, however statisticians have hardly considered these groups in the construction of appropriate models and in the analysis of observations.

An *automorphism* of an incidence structure is a one-to-one map Φ of points onto points and blocks onto blocks such that $\Phi P = P$, $\Phi \mathcal{B} = \mathcal{B}$ and incidences are preserved. Thus, for our block design D with treatments $\{t_i\}$ and blocks $\{B_j\}$, an automorphism of the design is a one-to-one map Φ such that (t_i, B_j) is a flag $(\in I)$ if and only if $(\Phi t_i, \Phi B_j)$ is a flag $(\in I)$. The set of all such automorphisms of an incidence structure (or a design) form a finite group, a *group* being a set of elements $G = \{a, b, c, \ldots\}$ having a binary operation called a product such that

(i) The set G is closed under the binary operation.

(ii) The elements of G are associative.

(iii) The set G has an identity element e.

(iv) Each element a in G has a unique inverse a^{-1}.

Such an automorphism group of a design can be represented by a group of permutations of the flags in the design. In fact, if G is the group of automorphisms of a design, each $g \in G$ may be represented as a permutation matrix $P(g)$ reflecting the permutation of the flags under g.

In general, structurally different BIB designs with identical parameters v, b, r, k, λ have different automorphism groups. We will thus use these automorphism groups to modify our usual linear model. We assume the distribution of $\underline{Y} = (Y_1, Y_2, \ldots, Y_n)'$ is such that its covariance matrix $\mathrm{Var}(\underline{Y}) = \Sigma$ is invariant under the action of the permutation matrices representing the group of automorphisms, i.e., $\sigma_{g(i)g(j)} = \sigma_{ij}$ for $i, j = 1, 2, \ldots, n$. This implies $P(g)\Sigma = \Sigma P(g)$ for all g in G, that is, Σ belongs to the *commutant algebra* of the group of matrices permuting the flags of the design. Thus, if t is the dimension and $V_1, V_2, \ldots, V_t$ is a linear basis of this algebra,

$$\Sigma = c_1V_1 + c_2V_2 + \ldots + c_tV_t.$$

It is of interest to determine the commutant algebra for a given design and then to use the knowledge that Σ belongs to the commutant algebra in the analysis of the experiment.

The first three chapters in this dissertation are introductory in nature; they provide motivation, preliminary formulations and a historical framework for these problems. In Chapter IV, for the designs based on the desarguesian finite projective plane PG(2,2) and the desarguesian finite affine plane EG(2,3), the matrix representations $\{P(g)\}$ of the automorphism groups of the designs are decomposed into direct sums of irreducible representations. The simple components of the corresponding commutant algebras are then derived.

In Chapter V, we consider designs based on desarguesian finite projective planes PG(2, p^r) and desarguesian finite affine planes EG(2, p^r). Using a theorem of Schur and a counting argument, the dimensions of the corresponding commutant algebras are found to be six and seven, respectively. For designs based on the projective geometries PG(k, p^r), $k \geq 3$, and affine geometries EG(k, p^r), $k \geq 3$, if the (k-1)-dimensional spaces in the geometries are taken to be blocks of the designs, the corresponding commutant algebras will have dimensions seven and eight, respectively. Appropriate bases for all these commutant algebras are indicated.

In Chapter VI, we consider the effect on the analysis of a BIB design if we assume the covariance matrix belongs to one of four possible algebras. We wish to estimate the variance components and to determine estimates of the treatment contrasts; both the fixed effects case and the random blocks situation are considered.

II. REVIEW OF LITERATURE

The occurrence and importance of symmetry in nature has often been considered. H. Weyl (1952), for example, studied the forms of symmetry and was led to the concept of invariance of spatial configurations under automorphic transformations. Upon encountering a "structure-endowed entity Σ," Weyl encourages an examination of the group of automorphisms of the entity in order to gain insight into its structure. It is only natural then to examine the structure of our BIB design by examining its group of automorphisms.

We will represent the group of automorphisms of a design in terms of permutation matrices reflecting the permutation of the flags in the design, and as mentioned earlier, we will be concerned with the commutant algebra of these permutation matrices. The *relationship algebra* of the BIB design, introduced by A. T. James (1957), is inherently useful. This algebra is generated by four n×n matrices based on symmetric relations between the experimental units: I, the n×n identity matrix; G, the n×n matrix of ones; $B = ((b_{ij}))$, $b_{ij} = 1$ if the experimental units i and j are in the same block, $b_{ij} = 0$ otherwise; and $T = ((t_{ij}))$, $t_{ij} = 1$ if the experimental units i and j share a treatment, $t_{ij} = 0$ otherwise. James stated that these four matrices will generate a non-commutative seven-dimensional associative algebra R_7 with basis matrices I, G, B, T, BT, TB and BTB. Since this algebra is generated by symmetric relations, it will be isomorphic to the direct sum of three one-

dimensional and one 2×2 complete matrix algebra. (Such algebraic notions will be discussed in Section 3.1 of Chapter III.) Mann (1960) later determined that if the design is *symmetric* , i.e., b = v, then BTB = $(r-\lambda)B + \lambda G$ and the (semi-simple) algebra $\mathcal{R}_6$ determined by the relationship matrices will be six-dimensional and non-commutative. In fact, the relationship algebra will be seven-dimensional if and only if the design is *asymmetric*, i.e., b>v. (It is known for a BIB design that b≥v.) Table 1 contains a multiplication table of the basis matrices of $\mathcal{R}_7$.

TABLE 1

I	G	B	T	BT	TB	BTB
G	nG	kG	rG	krG	krG	k^2rG
B	kG	kB	BT	kBT	BTB	kBTB
T	rG	TB	rT	$\lambda G + (r-\lambda)T$	rTB	$\lambda kG + (r-\lambda)TB$
BT	krG	BTB	rBT	$\lambda kG + (r-\lambda)BT$	rBTB	$\lambda k^2G + (r-\lambda)BTB$
TB	krG	kTB	$\lambda G + (r-\lambda)T$	$\lambda kG + (r-\lambda)kT$	$\lambda kG + (r-\lambda)TB$	$\lambda k^2G + (r-\lambda)kTB$
BTB	k^2rG	kBTB	$\lambda kG + (r-\lambda)BT$	$\lambda k^2G + (r-\lambda)kBT$	$\lambda k^2G + (r-\lambda)BTB$	$\lambda k^3G + k(r-\lambda)BTB$

James pointed out that important properties of the experimental design are indicated by examining the decomposition of the semi-simple algebra $\mathcal{R}_7$ into the direct sum of complete matrix algebras. Equivalently, the algebra will decompose into a direct sum of minimum two-sided ideals, each with a corresponding principal idempotent. The sums of squares in

the usual analysis of variance are quadratic forms of these idempotents. Mann also considered the decomposition of R_6 and R_7 into complete matrix algebras, using the fact that the relationship algebra is just the algebra generated by D'D, L'L and I, where D' and L' are the matrices found in the linear model.

These relationship algebras will also be of value in a different sense: the four generating matrices are invariant under the action of the automorphism group of the BIB design, i.e., P(g)IP(g)' = I, P(g)GP(g)' = G, P(g)BP(g)' = B and P(g)TP(g)' = T for all g in G, the automorphism group. Each of the generating matrices thus belongs to the commutant algebra of the group of matrices permuting the flags of the design. Hence, the relationship algebra for the design will be a subalgebra of this commutant algebra. In fact, James states, "For certain designs, the relationship algebra is the commutator (*commutant*) algebra of the representation of a group expressing the symmetry of the experimental design. Such will be the subject of a further paper." James, however, published no further work on this particular subject.

Our present work is in the spirit of Hannan (1965) and McLaren (1963). Hannan considered the structures of the commutant algebras of given group representations, in particular, the four-dimensional algebra of the randomized block design (generated by the matrices I, G, B and T) which coincides with the commutant algebra of the "representation of the group of symmetries." He also indicated that the pairs of flags can be divided into equivalence classes, where two flag pairs (s, t) and (s', t') are in the same class if there is a g in the group of automorphisms G such that gs = s' and gt = t'. If $\chi^i(s, t)$ is 1

when (s, t) belongs to the i^{th} equivalence class and 0 otherwise, then for any $\Sigma = ((\sigma_{st}))$ in the commutant algebra of a design, $\sigma_{st} = \sum \alpha_i \chi^i(s, t)$, i.e., $\sigma_{st} = \sigma_{s't'}$ if the flag pairs (s, t) and (s', t') are in the same equivalence class. With the aid of representation theory, McLaren also considered the algebra of matrices commuting with a matrix group representation. He studied the decomposition of the representation into a direct sum of irreducible representations and used this to determine the structure of a matrix in the commutant algebra and conditions under which the matrix may be real and symmetric.

In a series of papers, Dubenko, Sysoev and Shaikin (1976, I, II), Sysoev and Shaikin (1976, I, II, III) and Sysoev (1977) also examined the structure of the commutant algebra for several designs by using groups of permutations based on "symmetry properties." For a BIB design, the *symmetry group* is defined to be the set of all g such that $b_{ij} = b_{g(i)g(j)}$ and $t_{ij} = t_{g(i)g(j)}$ for all experimental unit pairs (flag pairs). Sysoev and Shaikin (1976, II) showed that in both the asymmetric and symmetric cases, the relationship algebra may coincide with the commutant algebra only if $\lambda = 1$. In fact, they state, without proof, "For symmetric schemes PG(m, q) the R-equivalence classes coincide with equivalence classes in the sense (23), i.e., the extended relationship algebra P_7 is a commutator algebra. In the special case of block-schemes PG(2, q) for which $\lambda = 1$, the commutator algebra coincides with the relationship algebra R_6. The relationship algebra R_7 is the commutator algebra in the case of asymmetric block-schemes with $\lambda = 1$, generated by the finite Euclidean geometries EG(2, q)." For the case $\lambda > 1$, they also extended R_6 by the addition of the matrix $S = ((s_{ij}))$

where $s_{ij} = 1$ if $(BT)_{ij} = (TB)_{ij} = 1$, 0 otherwise. S is invariant under the action of the automorphism group of the BIB design since BT and TB are invariant. It is also linearly independent of the matrices in the relationship algebra if and only if $\lambda > 1$.

Sysoev (1977) showed that in the asymmetric case for $\lambda > 1$, R_7 could be extended by the addition of S. The linear space of the matrices may not necessarily constitute an algebra; if it does, however, it will be a subalgebra of the commutant algebra containing the appropriate relationship algebra. Sysoev found that certain designs exist in which the linear space is an algebra; these designs are characterized by the following block intersection properties:

(i) Each pair of different blocks share either δ or 0 treatments (true for all symmetric designs).

(ii) Each triple of blocks share either δ, ν or 0 treatments $(0 \leq \nu < \delta)$.

(iii) If any two distinct blocks share δ treatments, then ρ blocks in the design share those δ treatments.

(iv) For any pair of treatments α, β and any pair of blocks B_i, B_j such that $\alpha \in B_i$, $\alpha \notin B_j$, $\beta \in B_j$, $\beta \notin B_i$ and $B_i \cap B_j \neq \emptyset$, there are σ blocks which share the treatments α and β and which share ν treatments with $B_i \cap B_j$.

For the designs based on the finite projective geometries PG(k, s), $k \geq 3$, and the finite affine geometries EG(k, s), $k \geq 3$, the addition of S to the appropriate relationship algebra results in an algebra.

Following the lead of James, Sysoev and Shaikin (1976, I, II) and Sysoev (1977) develop idempotent base matrices for the algebras R_6 and R_7 and the extended relationship algebras P_7 and P_8. For example, in the case of R_7, the base matrices are

$$E_0 = \frac{BT}{r-\lambda} - \frac{\phi+1}{bk} G \qquad E_1 = \frac{TB}{r-\lambda} - \frac{\phi+1}{bk} G$$

(2.1)
$$E_2 = \frac{BTB}{k(r-\lambda)} - \frac{\phi+1}{bk} G \qquad E_3 = \frac{T}{r} - \frac{G}{bk}$$

$$E_4 = \frac{G}{bk} \qquad E_5 = I - \frac{B}{k} - \frac{\phi+1}{\phi r} T + \frac{BT + TB}{\phi(r-\lambda)} - \frac{BTB}{\phi k(r-\lambda)}$$

$$E_6 = \frac{B}{k} - \frac{BTB}{k(r-\lambda)} + \frac{\phi}{bk} G \qquad \phi+1 = \frac{kr}{r-\lambda}$$

A multiplication table for these matrices is found in Table 2. Base matrices for R_6 may also be derived from (2.1) by setting $E_6 = 0$ and $BTB = (r - \lambda)B + \lambda G$.

TABLE 2

	E_0	E_1	E_2	E_3	E_4	E_5	E_6
E_0	E_0	$(\phi+1)E_2$	E_2	E_0	0	0	0
E_1	$(\phi+1)E_3$	E_1	E_1	E_3	0	0	0
E_2	E_0	E_2	E_2	$\frac{E_0}{\phi+1}$	0	0	0
E_3	E_3	E_1	$\frac{E_1}{\phi+1}$	E_3	0	0	0
E_4	0	0	0	0	E_4	0	0
E_5	0	0	0	0	0	E_5	0
E_6	0	0	0	0	0	0	E_6

In addition, the authors give necessary and sufficient conditions for matrices in the four algebras to be non-singular, positive definite symmetric and non-negative definite symmetric. For example, in the case of R_7, necessary and sufficient conditions for the matrix $V = \sum_{i=0}^{6} c_i E_i$ to be symmetric positive definite are:

$$c_0 = c_1$$

$$c_i > 0 \quad i = 2, 3, \ldots, 6$$

$$c_2 c_3 - (\phi+1) c_1^2 > 0$$

For R_6, the above conditions hold except for the condition on c_6 since E_6 is now 0.

It is of interest to note here that in the sums of squares in the usual analysis of variance, $-\frac{1}{\phi}(E_0+E_1) + \frac{1}{\phi} E_2 + \frac{\phi+1}{\phi} E_3$ is the idempotent for the treatments (eliminating blocks) component, E_2 is the idempotent for the treatments part of the blocks (ignoring treatments) component, E_4 is the idempotent for the grand mean, E_5 is the idempotent for the intra-block error and E_6 is the idempotent for the remainder part of the blocks (ignoring treatment) component. We see this to be true by first considering the observation vector $\underline{Y} = (Y_1, Y_2, \ldots, Y_n)'$ and defining

$g_i = i^{th}$ treatment total; $i = 1, \ldots, v$

$h_j = j^{th}$ block total; $j = 1, \ldots, b$

T_i = sum of block totals in which the i^{th} treatment occurs; $i = 1, \ldots, v$

(as in Scheffe (1959)). Note that $B\underline{Y}$ is an n-dimensional vector such that Y_j is replaced by h_j, $T\underline{Y}$ is an n-dimensional vector such that Y_i

is replaced by g_i and TBY is a vector such that Y_i is replaced by T_i. Thus

$$\underline{Y}'I\underline{Y} = \sum_{i=1}^{n} Y_i^2$$

$$\underline{Y}'G\underline{Y} = \left(\sum_{i=1}^{n} Y_i\right)^2$$

$$\underline{Y}'T\underline{Y} = \sum_{i=1}^{v} g_i^2$$

$$\underline{Y}'B\underline{Y} = \sum_{j=1}^{b} h_j^2$$

$$\underline{Y}'BT\underline{Y} = \underline{Y}'TB\underline{Y} = \sum_{i=1}^{v} g_i T_i$$

$$\underline{Y}'BTB\underline{Y} = \sum_{i=1}^{v} T_i^2$$

Therefore,

$$\underline{Y}' \left(-\frac{1}{\phi}(E_0 + E_1) + \frac{1}{\phi}E_2 + \frac{\phi + 1}{\phi}E_3\right) \underline{Y}$$

$$= \frac{1}{\phi}\underline{Y}' \left(-\frac{BT}{r - \lambda} - \frac{TB}{r - \lambda} + \frac{BTB}{k(r - \lambda)} + \frac{kT}{r - \lambda}\right) \underline{Y}$$

$$= \frac{1}{\phi}\left(\frac{k}{r - \lambda}\sum_{i=1=}^{v} g_i^2 - \frac{2}{r - \lambda}\sum_{i=1}^{v} g_i T_i + \frac{1}{k(r - \lambda)}\sum_{i=1}^{v} T_i^2\right)$$

$$= \frac{k}{\lambda v}\sum_{i=1}^{v} (g_i - k^{-1}T_i)^2 = SS_{tr.el.bl.}$$

$$\underline{Y}'E_2\underline{Y} = \frac{1}{k(r - \lambda)}\sum_{i=1}^{v} T_i^2 - \frac{\phi + 1}{n}\left(\sum_{i=1}^{n} Y_i\right)^2$$

$$\underline{Y}'E_4\underline{Y} = \frac{1}{n}\left(\sum_{i=1}^{n} Y_i\right)^2$$

$$\underline{Y}'E_5\underline{Y} = \sum_{i=1}^{n} Y_i^2 - \frac{1}{k}\sum_{j=1}^{b} h_j^2 - \frac{\phi + 1}{\phi r}\sum_{i=1}^{v} g_i^2 + \frac{2}{\phi(r - \lambda)}\sum_{i=1}^{v} g_i T_i$$

$$- \frac{1}{\phi k(r - \lambda)}\sum_{i=1}^{n} T_i^2 = \sum_{i=1}^{n} Y_i^2 - \frac{1}{k}\sum_{j=1}^{b} h_j^2 - SS_{tr.el.bl}$$

$$= SS_{error}$$

$$\underline{Y}'E_6\underline{Y} = \frac{1}{k}\sum_{j=1}^{b} h_j^2 - \frac{1}{k(r - \lambda)}\sum_{i=1}^{v} T_i^2 + \frac{\phi}{n}\left(\sum_{i=1}^{n} Y_i\right)^2.$$

We now concern ourselves with the analysis of balanced incomplete block designs; Scheffe (1959), Cochran and Cox (1950) and Kempthorne (1952) are standard references for the traditional analysis. Appendix A contains a short discussion of the estimation of treatment contrasts and the analysis of variance for these designs. For this discussion, either a normal fixed effects model $E(\underline{Y}) = \mu\underline{j}_n + D'\underline{\tau} + L'\underline{\beta}$, $Var(\underline{Y}) = \sigma^2 I_n$ or a normal random block model $E(\underline{Y}) = \mu\underline{j}_n + D'\underline{\tau}$, $Var(\underline{Y}) = \sigma^2 I_n + \sigma_1^2 L'L$ is assumed. In this dissertation, however, we are assuming that the covariance matrix has a special linear structure. Consequently, we now concern ourselves with estimation of the variance-covariance components and estimation of the treatment parameters under the assumption of our new model.

Regarding the estimation of our treatment, and perhaps block parameters, it is of interest to determine when the *best linear unbiased*

estimator (b.l.u.e.) of a parametric function is identical to that acquired under the standard analysis. We thus want to find if a b.l.u.e. of the linearly estimable parametric function $\sum_{i=1}^{u} \rho_i \theta_i$ can be found through a solution of the normal equations, the solution not depending on the covariance structure. Many have dealt with such parametric estimation, among them Kruskal (1968), Watson (1972), Mitra and Moore (1973), Mitra and Thorne (1976) and Zyskind (1975). The efficiency of the simple least squares estimator with respect to the best linear unbiased estimator has also been discussed by Watson (1967) and Baksalary and Kala (1980). (The simple least squares estimates are those which are determined under the assumption $\text{Var}(\underline{Y}) = \sigma^2 I$; if $E(\underline{Y}) = X\underline{\theta}$, the simple least squares estimate of $\underline{\theta}$ is derived from a solution of the normal equation $(X'X)\underline{\theta} = X'Y$. If $\text{Var}(\underline{Y})$ is a non-singular matrix V, the b.l.u.e. of $\underline{\theta}$ is derived from the normal equation $(X'V^{-1}X)\underline{\theta} = X'V^{-1}\underline{Y}$.)

Research has also been plentiful in the field of estimation of covariance matrices which are linear combinations of given symmetric matrices. Many authors have dealt with unbiased estimation of functions of the parameters in a covariance matrix, functions $\sum_{i=1}^{t} \nu_i c_i$ where $V = c_1V_1 + c_2V_2 + \ldots + c_tV_t$ is the appropriate covariance matrix. Seely (1969, 1971) in particular wrote much on the unbiased and minimum variance unbiased estimation of a function $\sum_{i=1}^{t} \nu_i c_i$ using quadratic $(\underline{Y}'A\underline{Y})$ or linear plus quadratic $(\underline{a}'\underline{Y} + \underline{Y}'A\underline{Y})$ estimators. Pukelsheim (1976) and Pukelsheim and Styan (1978) studied variance component coefficients by determining a derived linear model where the variance component coefficients comprise the mean vector of the derived model.

Rao (1973) considered variance components by examining MINQUE estimators, minimum norm invariant quadratic unbiased estimators. Gnot, Klonecki and Zmyślony (1977), Kleffé (1977) and Drygas (1977) also dealt with estimation of the variance components; several classes of estimators were examined by them.

In particular, Sysoev and Shaikin (1976, III) found sufficient statistics for the coefficients of a covariance matrix under the assumption that $\underline{Y}$ is normally distributed with mean $\underline{0}$ and covariance matrix belonging to one of the three algebras R_6, R_7 or P_7. For example, if the covariance matrix V is in R_7, i.e., $V = \sum_{i=0}^{6} c_i E_i$ and the set $\{\underline{Y}_\mu, u = 1, 2, \ldots, N\}$ represents the repeated sample vectors, unbiased estimators of $c_0, c_1, \ldots, c_6$ are

$$\hat{c}_0 = \hat{c}_1 = \frac{\phi + 1}{\phi^2(v-1)} \operatorname{tr} \left(\left[\frac{\phi + 2}{2(\phi+1)} (E_0 + E_1) - E_2 - E_3 \right] \sum_{\mu=1}^{N} \underline{Y}_\mu \underline{Y}_\mu' \right)$$

$$\hat{c}_2 = \frac{\phi + 1}{\phi^2(v-1)} \operatorname{tr} \left(\left[-(E_0 + E_1) + (\phi + 1)E_2 + E_3 \right] \sum_{\mu=1}^{N} \underline{Y}_\mu \underline{Y}_\mu' \right)$$

$$\hat{c}_3 = \frac{\phi + 1}{\phi^2(v-1)} \operatorname{tr} \left(\left[-(E_0 + E_1) + E_2 + (\phi + 1)E_3 \right] \sum_{\mu=1}^{N} \underline{Y}_\mu \underline{Y}_\mu' \right) \tag{2.2}$$

$$\hat{c}_4 = \operatorname{tr} \left(E_4 \sum_{\mu=1}^{N} \underline{Y}_\mu \underline{Y}_\mu' \right), \quad \hat{c}_5 = \frac{1}{bk-b-v+1} \operatorname{tr} \left(E_0 \sum_{\mu=1}^{N} \underline{Y}_\mu \underline{Y}_\mu' \right)$$

$$\hat{c}_6 = \frac{1}{b-v} \operatorname{tr} \left(E_6 \sum_{\mu=1}^{N} \underline{Y}_\mu \underline{Y}_\mu' \right), \quad (\phi + 1) = \frac{kr}{r - \lambda}.$$

(Minor modifications are necessary for covariance matrices in either R_6 or P_7.) Sysoev (1977) also considered the case where the covariance matrix is in the algebra P_8.

III. PRELIMINARY FORMULATIONS

3.1 General Notation, Definitions and Properties

We shall be using terminology and properties from several mathematical fields, among them linear algebra and group theory. We assume a basic knowledge of the theory of linear spaces (vector spaces); we present here only the notation we use and simple definitions (see, for example, Halmos (1958)). If U and V are subspaces in a vector space W, then $U + V$ is the set $\{\underline{u} + \underline{v} \; ; \; \underline{u} \in U, \; \underline{v} \in V\}$; if U and V are disjoint, i.e., $U \cap V = \{\underline{0}\}$, we write the *direct sum* $U \oplus V$ instead of $U + V$. For a subset U of a vector space, the *span* of U, written sp U, is the set of all finite linear combinations of elements in U. We shall define an *inner product space* to be a vector space W with a numerically valued function of the ordered pair of vectors, denoted the *inner product* $\langle \underline{x}, \underline{y} \rangle$, such that

(i) $\langle \underline{x}, \underline{y} \rangle = \langle \underline{y}, \underline{x} \rangle$

(ii) $\langle \alpha \underline{x}_1 + \beta \underline{x}_2, \underline{y} \rangle = \alpha \langle \underline{x}_1, \underline{y} \rangle + \beta \langle \underline{x}_2, \underline{y} \rangle$

(iii) $\langle \underline{x}, \underline{x} \rangle > 0$ if $\underline{x} \neq 0$.

For a subspace U of a vector space W, the set $U^{\perp}$ denotes the set of all vectors $\underline{v} \in W$ such that $\langle \underline{u}, \underline{v} \rangle = 0$ for all $\underline{u} \in U$, i.e., $U^{\perp}$ is the *orthogonal complement* of the subspace U.

Now let A be a linear operator on a vector space U into V. The *range* of A, the set $AU = \{A\underline{u} : \underline{u} \in U\}$, is denoted $R(A)$; the *null space* of A, the set $\{\underline{u} \in U : A\underline{u} = 0\}$, is denoted $N(A)$. The *rank* of A, indicated by $\rho(A)$, is the dimension of $R(A)$; the *nullity* of A, denoted $\nu(A)$, is the dimension of $N(A)$. For two square matrices A and B of degrees m and n, respectively, we define the *Kronecker product (tensor or direct product)* of A and B to be the $mn \times mn$ matrix $A \otimes B = ((a_{ij}B))$. We shall also denote the *direct sum* of the matrices $A_1, A_2, \ldots, A_n$ as $A = \text{diag}\,(A_1, A_2, \ldots, A_n) = \sum_{i=1}^{n} \oplus A_i = A_1 \oplus A_2 \oplus \ldots \oplus A_n$. A' will denote the *transpose* of a matrix; A^{-1} will indicate the *inverse*.

For the statistician who does not have a ready knowledge of mathematical structure theory, we introduce several definitions for terms used throughout this dissertation (see Gleeson and McGilchrist (1978) or Van der Waerden (1970)). A *ring* A is a non-empty set R equipped with two binary operations (addition and multiplication) such that:

(i) R is an abelian (commutative) group under addition with a unit (zero) element.

(ii) R is associative with respect to the operation of multiplication.

(iii) The distributive laws of multiplication hold.

A *division ring (skew field)* is a ring with a multiplicative identity 1 and a multiplicative inverse for every non-zero element. A *field* is a division ring in which multiplication is commutative. A field which contains a root for every equation $\sum_{i=0} a_i x^i = 0$ with coefficients in that field is said to be *algebraically closed*; clearly the complex numbers $\mathbb{C}$

have this property, while the real numbers $\mathbb{R}$ do not. An *algebra* M is a ring which is also a finite-dimensional vector space over a field F and which satisfies the condition that $\alpha(\underline{u}\underline{v}) = \underline{u}(\alpha\underline{v}) = (\alpha\underline{u})\underline{v}$ for all $\alpha \in F$, $\underline{u}$ and $\underline{v} \in M$. A *complete matrix algebra* is the full algebra of matrices of a given dimension.

We have already mentioned ideals and their corresponding idempotents. We now define a *right (left) ideal* to be a non-empty subset A of a ring R such that

(i) A is an additive subgroup of R.

(ii) For every $r \in R$ and $a \in A$, then $ar \in A$ ($ra \in A$).

A *two-sided ideal* is a right ideal which is also a left ideal. A right (left, two-sided) ideal is *minimal* if there is no right (left, two-sided) ideal B in R such that $\{0\} \subset B \subset A$. The ideal is *nilpotent* if $A^n = \{0\}$ for some integer n. The *descending chain condition* of right ideals holds for R if every strictly decreasing sequence of right ideals $A_1 \supset A_2 \supset \ldots$ has only a finite number of right ideals. A *semi-simple ring* is a ring in which the descending chain condition for right ideals holds and in which there are no non-zero nilpotent right ideals. We do know that a semi-simple ring is isomorphic to a direct sum of complete matrix rings (Wedderburn-Artin Theorem). From Theorem 4.6 in Gleeson and McGilchrist (1978), the ring generated by a finite set of n×n symmetric matrices is semi-simple if one of the generating matrices is I. Therefore, each of our relationship algebras and extended relationship algebras is isomorphic to a direct sum of complete matrix rings. The *principal idempotents* are the identity elements of the ideals.

We now move to a few group theoretic notions (see Hall (1959) or Carmichael (1937)). We have previously mentioned permutations; we now formally state that a *permutation* is a one-to-one mapping of a set X onto itself. Of course, in this dissertation, we are concerned only with a finite set X. We shall use cyclic notation to indicate these permutations, e.g., for the mapping

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix},$$

we write (34). (We will omit cycles of length one.) A *permutation group* is a group whose elements are permutations of a finite set X; the group has *degree n* if X has n symbols. Such a permutation group G is *transitive* if for every pair x_i and x_j in X, there exists a $g \in G$ such that $gx_i = x_j$; if the group is *intransitive*, it will divide X into transitive sets (*orbits*). The group G is *k-ply transitive* if any ordered set of k different symbols of X is taken into an arbitrary ordered set of k different symbols of X by some g in G. Two elements g_i and g_j of G are *conjugate* if there exists a $g \in G$ such that $g_i = gg_jg^{-1}$; this defines an equivalence relation on G for which the equivalence classes are called *conjugacy classes*. Finally, we let G_x denote the set of all elements in G which fix the element x, i.e., $G_x = \{g \in G : gx = x\}$. Such a set is a subgroup of the group G, called the *stabilizer* of the element x (*isotropy group* at x). For a transitive group G, the *order* or size of G_x is the order of G divided by its degree.

A theorem from Schur (1933) is particularly useful in our work. We quote from Wielandt (1964, p. 80):

> Theorem 3.1 (Theorem 28.4) If a transitive permutation group G is regarded as a matrix group G^*, then the matrices which commute with all the matrices of G^* form a ring $V = V(G)$. We call V the centralizer ring corresponding to G. V is a vector space over the complex number field which has the matrices $B(\Delta)$ corresponding to the orbits Δ of G_1 as a linear basis. In particular, the dimension of V coincides with the number k of orbits of G_1.

The matrix $B(\Delta) = ((v_{ij}^{\Delta}))$, i, j = 1, 2, ..., n associated with the orbit Δ is defined as:

$v_{ij}^{\Delta} = 1$ if there exists $g \in G_1$, $\delta \in \Delta$ such that $g1 = j$, $g\delta = i$

$= 0$ otherwise.

From a lemma by Burnside (see, for example, Liu (1968)), if h is the number of orbits of G_1,

$$h = \frac{1}{|G_1|} \sum_{g \in G_1} \Psi(g) \tag{3.1.1}$$

where $\Psi(g)$ is the number of elements in the set X which are invariant under the permutation g and $|G_1|$ is the order of the stabilizer subgroup G_1.

3.2 Definitions and Results from Group Representation and Character Theory

Group representation theory is used throughout this dissertation; since it is a subject with which many statisticians are not familiar, we will present some important definitions and results (see, for example, Serre (1977) or Ledermann (1977)). First consider a finite group G with

order $\gamma = |G|$; let C_i of order γ_i, $i = 1, 2, \ldots, r$, be the conjugacy classes of G and let C_i' be the inverse class of C_i. We call a *representation* of G a homomorphic mapping of the group G into a group of isomorphisms GL(V) of a vector space V onto itself. We will be concerned with representations $A : G \to GL_n(K)$ where $GL_n(K)$ is the general linear group of degree n with coefficients in a ground field K. Thus, we have a *matrix representation of degree n over* K such that for every $g \in G$, there exists a non-singular matrix A(g) of degree n with coefficients in K, where $A(g)A(h) = A(gh)$ for every $g, h \in G$.

Two matrix representations $\{A(g)\}$ and $\{B(g)\}$ of G are *equivalent* if there is a fixed non-singular matrix T with coefficients in K such that $A(g) = T^{-1}B(g)T$ for all $g \in G$; otherwise, they are *inequivalent*. If $\{A(g)\}$ and $\{B(g)\}$ are equivalent, we write $\{A(g)\} \sim \{B(g)\}$. A matrix representation $\{A(g)\}$ is *irreducible* if there is no non-singular matrix T over K such that

$$T^{-1}A(g)T = \begin{pmatrix} B(g) & 0 \\ D(g) & C(g) \end{pmatrix}$$

for all $g \in G$. A matrix representation $\{A(g)\}$ is *completely reducible* if $\{A(g)\} \sim \{\sum_{\lambda=1}^{j} \oplus A_\lambda(g)\}$, where $\{A_\lambda(g)\}$ $(\lambda = 1, 2, \ldots, j)$ are irreducible matrix representations of G over K. By Maschke's Theorem, if K is a field of characteristic zero or prime to γ, then every matrix representation of G over K is completely reducible. (A field of characteristic zero is a field which has a subfield isomorphic to the rationals. Hence, if K is the field of complex numbers $\mathbb{C}$ or the field of real numbers $\mathbb{R}$, a matrix representation of G over K is completely reducible.

Since we are concerned only with complex and real fields in this dissertation, we will consider our representations as completely reducible.) For $K = \mathbb{C}$, the number of inequivalent, irreducible matrix representations of G is equal to the number of conjugacy classes in G, here taken to be r. Any given matrix representation $\{A(g)\}$ is then the direct sum of some (or all) of these irreducible representations $\{A_\lambda(g)\}$, $\lambda = 1, 2, \ldots, r$ (possibly repeated). It is also of interest to note that each of these representations is equivalent to a unitary representation.

A lemma by Schur is very useful in determining the dimension of the commutant algebra of a matrix representation.

Schur's Lemma. Let $\{A_\lambda(G)\}$ and $\{A_\mu(g)\}$ be two irreducible matrix representations of G over K and let P be a matrix over K such that

$$PA_\lambda(g) = A_\mu(g)P$$

for all g in G. Then (i) $P = 0$ or (ii) $\{A_\lambda(g)\}$ and $\{A_\mu(g)\}$ are equivalent (and P is non-singular). If $\{A_\lambda(g)\}$ and $\{A_\mu(g)\}$ are equivalent and the field K is algebraically closed, then P is a scalar multiple of the identity, i.e., $P = cI$ for a scalar c.

Unless otherwise specified, we now consider K to be the field $\mathbb{C}$.

To extend this lemma, first consider $\{A(g)\}$ as a completely reducible representation of G over the algebraically closed field $\mathbb{C}$. Then $\{A(g)\} \sim \{\sum_{\lambda=1}^{r} \oplus I_{d_\lambda} \otimes A_\lambda(g)\}$ where $\{A_\lambda(g)\}$, $\lambda = 1, 2, \ldots, r$ are the r inequivalent, irreducible representations of G such that $\{A_\lambda(g)\}$ has degree n_λ and occurs with multiplicity d_λ. If we assume the representation $\{A_\lambda(g)\}$ is of the form $\{\sum_{\lambda=1}^{r} \oplus I_{d_\lambda} \otimes A_\lambda(g)\}$, by an extension

of Schur's Lemma, the commutant algebra of $\{A(g)\}$, denoted $C(A)$, is

$$C(A) = \sum_{\lambda=1}^{r} \oplus M_{d_\lambda} \otimes I_{n_\lambda} \tag{3.2.1}$$

where M_{d_λ} is the complete matrix algebra over $\mathbb{C}$. Thus the dimension of the commutant algebra $C(A)$ is

$$\dim C(A) = \sum_{\lambda=1}^{r} d_\lambda^2. \tag{3.2.2}$$

Of course, the degree of the representation $\{A(g)\}$ is $n = \sum_{\lambda=1}^{r} d_\lambda n_\lambda$.

These multiplicities d_λ can be found with the use of characters. For a matrix representation $\{A(g)\}$, the *character* is defined as $\chi(g) = \text{Trace } A(g)$. The character is a class function over the conjugacy classes of G, i.e., $\chi(g) = \chi(h)$ if g and h are in the same conjugacy class; characters are also identical over equivalent representations. We shall thus write $\chi_\lambda(C_i)$, $i = 1, 2, \ldots, r$ for the characters of $\{A_\lambda(g)\}$, $\lambda = 1, 2, \ldots, r$. The degree of $\{A_\lambda(g)\}$ is $n_\lambda = \chi_\lambda(e)$, where e is the identity element of G. If $\{A_\lambda^*(g)\}$ is the complex conjugate of $\{A_\lambda(g)\}$ with corresponding character $\chi_\lambda^*(C_i)$, then $\chi_\lambda^*(C_i) = \overline{\chi}_\lambda(C_i)$ $= \chi_\lambda(C_i')$ where $\overline{\chi}_\lambda(C_i)$ is the complex conjugate of $\chi_\lambda(C_i)$. If $\phi(g)$ and $\psi(g)$ are two functions on G with values in C, we define an inner product $\langle \phi, \psi \rangle_G = \frac{1}{\gamma} \sum_{g \in G} \phi(g)\psi(g^{-1})$. Then if $\chi(C_i)$, $i = 1, 2, \ldots, r$ denote the characters of $\{A(g)\}$ and d_λ denotes the number of times $\{A_\lambda(g)\}$ appears in $\{A(g)\}$, we have

$$d_\lambda = \langle \chi | \chi_\lambda \rangle_G = \frac{1}{\gamma} \sum_{i=1}^{r} \gamma_i \overline{\chi}_\lambda(C_i)\chi(C_i). \tag{3.2.3}$$

Clearly, the characters for the inequivalent, irreducible representations

must be known in order to use this result. Character tables for many groups have been determined (see, for example, Littlewood (1950)).

Let the group G of permutations on n symbols act transitively on X and let $\{A(g)\}$ be a permutation matrix representation of degree n. Now consider the stabilizer H of the symbol x. If η_i elements of H belong to the class C_i, i = 1, 2, ..., r, by the Frobenius Reciprocity Theorem,

$$(3.2.4) \qquad d_\lambda = \langle 1 | \chi_\lambda \rangle_H = \frac{1}{\eta} \sum_{i=1}^{r} \eta_i \overline{\chi_\lambda}(C_i),$$

where $\sum_{i=1}^{r} \eta_i = \eta$ is the order of H (see McLaren (1963)). (Of course, if $\{A(g)\}$ is intransitive, it is the direct sum of transitive representations, so an extension of (3.2.4) is necessary.)

The irreducible representations $\{A_\lambda(g)\}$ (λ = 1, 2, ..., r) are classified as follows:

(i) $\{A_\lambda(g)\}$ is of the *first kind* if it is equivalent to a set of real matrices (the characters are real).

(ii) $\{A_\lambda(g)\}$ is of the *second kind* if it is equivalent to the complex conjugate representation $\{A_\lambda^*(g)\}$ but not to a set of real matrices (the characters are real).

(iii) $\{A_\lambda(g)\}$ is of the *third kind* if $\{A_\lambda(g)\}$ and $\{A_\lambda^*(g)\}$ are not equivalent (the characters are not all real).

From a test of Wigner,

$$(3.2.5) \qquad \varepsilon_\lambda = \frac{1}{\gamma} \sum_{g \in G} \chi_\lambda(g^2)$$

takes on the values 1, -1 or 0 if and only if $\{A_\lambda(g)\}$ is of the first, second or third kind, respectively. We have mentioned earlier (3.2.2)

that dim $C(A) = \sum_{\lambda=1}^{r} d_\lambda^2$; the dimension ℓ of the space of real symmetric matrices in $C(A)$ is

$$\ell = \frac{1}{2}\left[\sum_{\lambda=1}^{r} d_\lambda^2 + \sum_{\lambda=1}^{r} \varepsilon_\lambda d_\lambda\right] \tag{3.2.6}$$

(McLaren (1963)).

3.3 Finite Desarguesian Geometries and Their Collineations

A *k-dimensional finite projective geometry* (see, for instance, Carmichael (1937)) is a finite set of points such that:

(i) There are one or more lines, each containing at least three points.

(ii) If A and B are distinct points, there is one and only one line containing A and B.

(iii) For any non-collinear points A, B and C, if a line ℓ not containing A, B or C intersects line AB and line AC, then it intersects line BC.

(iv) If m is an integer less than k, all the points in the geometry are not in the same m-space (*m-flat*).

(v) The set of points cannot form a (k + 1)-space.

A *projective plane* is a two-dimensional projective geometry. The points of a projective geometry denoted as PG(k, s) may be represented as ordered (k + 1)-tuples $\underline{x} = (x_0, x_1, \ldots, x_k)'$ where $x_0, x_1, \ldots, x_k$ are elements of the Galois field GF(s), $s = p^r$ such that p is a prime and r is an integer. The elements $x_0, x_1, \ldots, x_k$ may not all be zero and $(x_0, x_1, \ldots, x_k) = (\mu x_0, \mu x_1, \ldots, \mu x_k)$ for all non-zero μ in GF(s). Similarly, a *hyperplane* ((k - 1) - space) is also defined by an ordered (k + 1)-tuple $\underline{y} = (y_0, y_1, \ldots, y_k)'$ where $y_0, y_1, \ldots, y_k$ are elements

of the GF(s), not all zero, such that $(y_0, y_1, \ldots, y_k)$ $= (\mu y_o, \mu y_1, \ldots, \mu y_k)$ for all non-zero μ in GF(s). A point $(x_o, x_1, \ldots, x_k)$ is contained in a hyperplane $(y_o, y_1, \ldots, y_k)$ if $x_0y_0 + x_1y_1 + \ldots + x_ky_k = 0$. If $k = 2$, a hyperplane is a line.

We used the notation PG(k, s) to indicate the projective geometry constructed by means of the Galois field. It will also designate any projective geometry which is isomorphic to a geometry constructed in this manner. For $k = 2$, these are precisely the *desarguesian* projective planes, i.e., projective planes in which the following *Theorem of Desargues* holds: If ABC and abc are two triangles in the same plane of a projective space such that Aa, Bb and Cc pass through the same point P, and if α = (Aa, Bb), β = (Aa, Cc) and γ = (Bb, Cc), then α, β and γ lie on the same line. For $k \geq 3$, every projective geometry is desarguesian and is isomorphic to a PG(k, s).

A k-dimensional *finite affine geometry EG(k, s)* can be obtained from PG(k, s) by omitting a hyperplane and all the points in that space. A point in this geometry can be written as an ordered k-tuple $\underline{x} = (x_1, x_2, \ldots, x_k)'$ where $x_1, x_2, \ldots, x_k$ are elements of GF(s). A hyperplane (for $k = 2$, a line) is represented as a $(k + 1)$-tuple $\underline{y} = (y_0, y_1, \ldots, y_k)'$ where $y_0, y_1, \ldots, y_k$ are elements in GF(s) such that not all of $y_1, y_2, \ldots, y_k$ equal zero and $(y_0, y_1, \ldots, y_k)$ $= (\mu y_0, \mu y_1, \ldots, \mu y_k)$ for all non-zero μ in GF(s). A point $(x_1, x_2, \ldots, x_k)$ is contained in a hyperplane $(y_0, y_1, \ldots, y_k)$ if $y_0 + x_1y_1 + x_2y_2 + \ldots + x_ky_k = 0$. A finite affine geometry is desarguesian if the corresponding projective geometry is desarguesian. Thus, EG(2, s) will refer to the desarguesian affine planes while EG(k, s), $k \geq 3$, will refer to all affine geometries of dimension $k \geq 3$.

These finite projective geometries PG(k, s) and finite affine geometries EG(k, s) are of interest since a balanced incomplete block design can be constructed from a finite geometry by taking points of the geometry as treatments and subspaces of the geometry as blocks. We shall primarily be concerned with the case where the blocks are taken to be the hyperplanes. Thus, for a projective geometry PG(k, s), the parameters for the design are:

$$
\begin{aligned}
v = b &= \frac{s^{k+1} - 1}{s - 1} \\
r = k &= \frac{s^{k} - 1}{s - 1} \qquad (3.3.1) \\
\lambda &= \frac{s^{k-1} - 1}{s - 1}
\end{aligned}
$$

For an affine geometry EG(k, s), the parameters for the design are:

$$
\begin{aligned}
v &= s^{k} & k &= s^{k-1} \\
b &= \frac{s^{k+1} - s}{s - 1} & \lambda &= \frac{s^{k-1} - 1}{s - 1} \qquad (3.3.2) \\
r &= \frac{s^{k} - 1}{s - 1}
\end{aligned}
$$

An automorphism of a geometry (and hence, our design) is called a *collineation*; customarily, the group of automorphisms is called the group of *collineations*. These collineations are such that the points, lines, and hence, the subspaces are permuted. Thus, a BIB design based on PG(k, s) or EG(k, s) will have as its full group of automorphisms the full collineation group PC(k, s) or EC(k, s) of the geometry. The full collineation group PC(k, s) of the finite projective geometry is repre-

sented by the homogeneous transformations of the points (and hence, the subspaces)

$$(3.3.3) \qquad \rho x_i^* = \sum_{j=0}^{k} a_{ij} x_j^{p^u} \quad (i = 0, 1, \ldots, k;\ u = 0, 1, \ldots, r-1),$$

where $\rho (\neq 0) \in GF(s)$ and the a_{ij} are the elements of $GF(s)$ such that

$$\Delta = \begin{vmatrix} a_{00} & a_{01} & \cdots & a_{0k} \\ a_{10} & a_{11} & \cdots & a_{1k} \\ & & \vdots & \\ a_{k0} & a_{k1} & \cdots & a_{kk} \end{vmatrix}$$

is non-zero. For a given value of p^u in (3.3.3), two transformations are identical if the coefficients a_{ij} are identical despite distinct values of ρ. The order of the group $PC(k, s)$ is

$$(3.3.4) \qquad \frac{r}{p^r - 1} \prod_{i=0}^{k} (p^{(k+1)r} - p^{ir}) = \frac{r}{s-1} \prod_{i=0}^{k} (s^{k+1} - s^i).$$

Considered as a permutation group on the points, this group is doubly transitive.

The following theorem also holds:

Theorem 3.2 For a finite projective geometry $PG(k, s)$, $s = p^r$, if the point vectors $\underline{x} = (x_0, x_1, \ldots, x_k)'$ are transformed by the collineation $A(x_0^m, x_1^m, \ldots, x_k^m)'$, $m = p^u$ for some $u = 0, 1, \ldots, r-1$, then the hyperplane vectors are transformed by the map

$$(A')^{-1} (y_0^m, y_1^m, \ldots, y_k^m)'.$$

Proof: First we note here that if $m = p^u$ for $u = 0, 1, \ldots, r - 1$ and if $a, b \in GF(s)$, then $(a + b)^m = a^m + b^m$. In the expansion of $(a + b)^m$, the coefficients of the terms other than a^m and b^m share the factor p; these coefficients are then 0 in GF(s).

A point $\underline{x}$ is contained in a hyperplane $\underline{y}$ if and only if $\underline{y}'\underline{x} = 0$. As $\underline{x}$ is transformed to $\underline{x}^*$, $\underline{y}$ will be transformed to a $\underline{y}^*$ such that $\underline{y}^{*\prime}\underline{x}^* = 0$. For a hyperplane $\underline{y}$ and any point $\underline{x}$ contained in it:

$$\underline{y}'\underline{x} = 0 \rightarrow y_0x_0 + y_1x_1 + \ldots + y_kx_k = 0$$

$$\rightarrow (y_0x_0 + y_1x_1 + \ldots + y_kx_k)^m = 0$$

$$\rightarrow y_0^m x_0^m + y_1^m x_1^m + \ldots + y_k^m x_k^m = 0$$

$$\rightarrow (y_0^m y_1^m \ldots y_k^m)\ (x_0^m x_1^m \ldots x_k^m)' = 0$$

$$\rightarrow (y_0^m y_1^m \ldots y_k^m)\ A^{-1}A\ (x_0^m x_1^m \ldots x_k^m)' = 0$$

All points $\underline{x}$ in the hyperplane $\underline{y}$ satisfy this last equation. This defines the hyperplane transformation. □

The set of transformations which fix the point $(1, 0, \ldots, 0)$ is the set of all collineations (3.3.3) such that

$$A(1^m, 0^m, \ldots, 0^m)' = \rho(1, 0, \ldots, 0)',\ \rho(\neq 0) \in GF(s).$$

As a result, $a_{00} = \rho$, $a_{10} = a_{20} = \ldots = a_{k0} = 0$. Omitting the equivalent transformations, we can say that the transformations which fix $(1, 0, \ldots, 0)$ are of the following form:

$$(3.3.5) \qquad A\underline{x}_m = \begin{pmatrix} 1 & a_{01} & a_{02} & \cdots & a_{0k} \\ 0 & a_{11} & a_{12} & \cdots & a_{1k} \\ & & & \vdots & \\ 0 & a_{k1} & a_{k2} & \cdots & a_{kk} \end{pmatrix} \begin{pmatrix} x_0^m \\ x_1^m \\ \vdots \\ x_k^m \end{pmatrix}.$$

Of these, the collineations which also fix the hyperplane $(0, 0, \ldots 0, 1)$ $(x_k = 0)$ are those which satisfy:

$$(A')^{-1} (0^m, 0^m, \ldots, 0^m, 1^m)' = \alpha(0, 0, \ldots, 0, 1)',$$

$$\alpha(\neq 0) \in GF(s),$$

which implies

$$\gamma(0, 0, \ldots, 0, 1)' = A'(1, 0, \ldots, 0, 1)', \ \gamma(\neq 0) \in GF(s).$$

Therefore, for the transformation in (3.3.5), $a_{k1} = a_{k2} = \ldots = a_{k,\,k-1} = 0$, while $a_{kk} = \gamma$. The collineations which fix the point $(1, 0, \ldots, 0)$ and the hyperplane $(0, 0, \ldots, 0, 1)$ are thus of the form:

$$(3.3.6) \qquad A\underline{x}_m = \begin{pmatrix} 1 & a_{01} & \cdots & a_{0,\,k-1} & a_{0k} \\ 0 & a_{11} & \cdots & a_{1,\,k-1} & a_{1k} \\ & & \vdots & & \\ 0 & a_{k-1,\,1} & \cdots & a_{k-1,\,k-1} & a_{k-1,\,k} \\ 0 & 0 & \cdots & 0 & a_{kk} \end{pmatrix} \begin{pmatrix} x_0^m \\ x_1^m \\ \vdots \\ x_k^m \end{pmatrix}$$

$a_{ij} \in GF(s)$; $m = p^u$; $u = 0, 1, \ldots, r-1$; such that A is non-singular (which implies a_{kk} is non-zero).

The full collineation group EC(k, s) of the desarguesian affine geometry EG(k, s) is represented by the non-homogeneous transformations

$$(3.3.7) \qquad x_i^* = a_{i0} + \sum_{j=1}^{k} a_{ij} x^{p^u} \quad (i = 1, 2, \ldots, k; \; u = 0, 1, \ldots, r-1; \; a_{ij} \in GF(s)),$$

such that

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ & & \vdots & \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{vmatrix}$$

is non-zero. The order of the group EC(k, s) is

$$(3.3.8) \qquad rp^{kr} \prod_{i=0}^{k-1} (p^{kr} - p^{ir}) = rs^k \prod_{i=0}^{k-1} (s^k - s^i).$$

Considered as a permutation group on the s^k points, EC(k, s) is doubly transitive.

We also have:

Theorem 3.3 For a finite affine geometry EG(k, s), if the point vectors $\underline{x} = (x_1, x_2, \ldots, x_k)'$ are transformed by the collineation

$$a + A\underline{x}_m = \begin{pmatrix} a_{10} \\ a_{20} \\ \vdots \\ a_{k0} \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ & & \vdots & \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{pmatrix} \begin{pmatrix} x_1^m \\ x_2^m \\ \vdots \\ x_k^m \end{pmatrix} \quad (a_{ij} \in GF(s); \; m = p^u; \; u = 0, 1, \ldots, r-1),$$

then the hyperplane vectors $(y_0, y_1, \ldots, y_k)$ are transformed as follows:

$$y_0^m - (y_1^m, y_2^m, \ldots, y_k^m)\, A^{-1}\underline{a} = \rho y_0^*;$$

$$(A')^{-1} \begin{pmatrix} y_1^m \\ y_2^m \\ \vdots \\ y_k^m \end{pmatrix} = \rho \begin{pmatrix} y_1^* \\ y_2^* \\ \vdots \\ y_k^* \end{pmatrix} ; \ \rho(\neq 0) \in GF(s).$$

Proof: Choose any hyperplane $\underline{y} = (y_0, y_1, \ldots, y_k)'$; for any point $\underline{x} = (x_1, x_2, \ldots, x_k)'$ in the hyperplane,

$$y_0 + y_1x_1 + \ldots + y_kx_k = 0$$

$$\rightarrow (y_0 + y_1x_1 + \ldots + y_kx_k)^m = 0$$

$$\rightarrow y_0^m + y_1^m x_1^m + \ldots + y_k^m x_k^m = 0$$

$$\rightarrow (y_0^m - (y_1^m, y_2^m, \ldots y_k^m)\, A^{-1}\underline{a}) +$$

$$+ (y_1^m, y_2^m, \ldots, y_k^m)A^{-1}(\underline{a} + A\underline{x}_m) = 0$$

This equation is satisfied for all points $\underline{x}$ on the hyperplane $\underline{y}$. Therefore, our transformed hyperplane is determined. □

The set of transformations which fix the point $(0, 0, \ldots, 0)$ is the set of collineations such that $\underline{a} + A(0^m, 0^m, \ldots, 0^m)'$ $= (0, 0, \ldots, 0)'$, where $\underline{a}$ and A are defined as in Theorem 3.3. This implies $\underline{a} = \underline{0}$. Of this set, the collineations which fix the hyperplane

$(0, 0, \ldots, 0, 1)$ $(x_k = 0)$ are those such that

$$A(x_1^m, x_2^m, \ldots, x_{k-1}^m, 0)' = (x_1^*, x_2^*, \ldots, x_{k-1}^*, 0)'.$$

Since $a_{k1}x_1^m + a_{k2}x_2^m + \ldots + a_{k,\,k-1}x_{k-1}^m = 0$ for all point vectors of the form $(x_1, x_2, \ldots, x_{k-1}, 0)$, then $a_{k1} = a_{k2} = \ldots = a_{k,\,k-1} = 0$. Thus, the collineations which fix the point $(0, 0, \ldots, 0)$ and the hyperplane $(0, 0, \ldots, 0, 1)$ are of the form:

$$(3.3.9)\quad Ax_{-m} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1,\,k-1} & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2,\,k-1} & a_{2k} \\ & & \vdots & & \\ a_{k-1,\,1} & a_{k-1,\,2} & \cdots & a_{k-1,\,k-1} & a_{k-1,\,k} \\ 0 & 0 & \cdots & 0 & a_{kk} \end{pmatrix} \begin{pmatrix} x_1^m \\ x_2^m \\ \vdots \\ x_k^m \end{pmatrix}$$

$$(m = p^u,\ u = 0, 1, \ldots, r - 1)$$

such that A is non singular with coefficients in GF(s) (which implies that a_{kk} is non-zero).

We also note that if the flags of a design are taken to be the incident point-hyperplane pairs, the groups PC(k, s) and EC(k, s) are transitive on the flags in their respective spaces. This is seen to be true from a result in Dembowski (1968, p. 80): "Suppose that $(r, \lambda) = 1$. If Γ is doubly point transitive, then Γ is flag transitive." (Γ here is the automorphism group of a design; (r, λ) indicates the greatest common factor of r and λ.) If the flags are the incident point-hyperplane pairs,

we see from (3.3.1) and (3.3.2) that

$$r = \frac{s^k - 1}{s - 1} = s^{k-1} + s^{k-2} + \ldots + 1$$

$$\lambda = \frac{s^{k-1} - 1}{s - 1} = s^{k-2} + s^{k-3} + \ldots + 1,$$

for the designs from both PG(k, s) and EG(k, s). Clearly $(r, \lambda) = 1$ and since PC(k, s) and EC(k, s) are doubly point transitive, they are flag transitive.

IV. DECOMPOSITION INTO A DIRECT SUM OF IRREDUCIBLE REPRESENTATIONS FOR PC(2, 2) AND EC(2, 3)

4.1 Introduction

By deriving the decomposition of the permutation representation of an automorphism group of a design into a direct sum of irreducible representations over an algebraically closed field, we may gain insight into the structure of the permutation representation and the corresponding commutant algebra. In this spirit, we therefore determine such decompositions for the permutation (matrix) representations of the full collineation groups PC(2, 2) and EC(2, 3) for the finite planes PG(2, 2) and EG(2, 3), respectively. We can then use the extension of Schur's lemma (3.2.1) to find the structure of the corresponding commutant algebra. From Wigner's test (3.2.5) we can show that the irreducible representations in the decomposition of the permutation representation of the groups PC(2, 2) and EC(2, 3) are all equivalent to real irreducible representations and hence to sets of real orthogonal matrices.

4.2 Decomposition of the Permutation Representation of PC(2, 2) into a Direct Sum of Irreducible Representations

We assign to the seven points of PG(2, 2) the letters A, B, C, D, E, F and G as follows:

x_0	1	0	0	1	1	0	1
x_1	0	1	0	1	0	1	1
x_2	0	0	1	0	1	1	1
	A	B	C	D	E	F	G

The lines of the plane are indicated by the columns in the following diagram:

$\underline{L_1}$	$\underline{L_2}$	$\underline{L_3}$	$\underline{L_4}$
$(x_0 = 0)$	$(x_1 = 0)$	$(x_2 = 0)$	$(x_0 + x_1 = 0)$
B	A	A	C
C	C	B	D
F	E	D	G

$\underline{L_5}$	$\underline{L_6}$	$\underline{L_7}$
$(x_0 + x_2 = 0)$	$(x_1 + x_2 = 0)$	$(x_0 + x_1 + x_2 = 0)$
B	A	D
E	F	E
G	G	F

The full collineation group PC(2, 2) of this plane consists of 168 3×3 non-singular matrices with coefficients in GF(2). Any given matrix A in this group induces a permutation of the points (and hence, the lines) through the linear transformation of the point vectors $A\underline{x}$. The group of permutations of the points, a subgroup of the symmetric group on seven symbols, is doubly transitive and has six conjugacy classes. Therefore, there will be six inequivalent irreducible representations of PC(2, 2). Table 3 contains the character table for this group (Littlewood (1950)). The cycles indicate the conjugacy classes

for the group and γ_i denotes the number of elements in the i^{th} conjugacy class; the irreducible representations are indexed by λ.

TABLE 3

Character Table of PC(2, 2), $|PC(2, 2)| = 168$

λ	cycle (conjugacy classes)	1^7	$1^3 2^2$	124	13^2	7	7
	order of i^{th} class γ_i	1	21	42	56	24	24
1		1	1	1	1	1	1
2		6	2	0	0	-1	-1
3		7	-1	-1	1	0	0
4		8	0	0	-1	1	1
5		3	-1	1	0	$\frac{1}{2}(-1 + i\sqrt{7})$	$\frac{1}{2}(-1 - i\sqrt{7})$
6		3	-1	1	0	$\frac{1}{2}(-1 - i\sqrt{7})$	$\frac{1}{2}(-1 + i\sqrt{7})$

The group of permutations on the 7 points induces a transitive group on the 21 flags (treatment-block combinations) of the design. Thus, the group of 21×21 matrices reflecting the permutations of the flags is a representation of the group PC(2, 2). In order to derive the decomposition of this representation into a direct sum of irreducible representations, we first consider the stabilizer H of the flag (A, L_3), i.e., the subgroup of PC(2, 2) which fixes the point A and line L_3. This subgroup has order $\frac{|PC(2, 2)|}{n} = \frac{168}{21} = 8$ and is induced by the following matrices (which transform the point vectors):

Matrices of H	Point Permutations	Cycles
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	identity	1^7
$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	(CE) (FG)	$1^3 2^2$
$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	(BD) (FG)	$1^3 2^2$
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$	(CF) (EG)	$1^3 2^2$
$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	(BD) (CE)	$1^3 2^2$
$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$	(BD) (CFEG)	124
$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$	(CG) (EF)	$1^3 2^2$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad (BD)(CGEF) \qquad 124$$

These matrices can be found from (3.3.6).

If n_i is the number of elements of H in the conjugacy class C_i of PC(2, 2), then clearly $n_1 = 1$, $n_2 = 5$, $n_3 = 2$ and $n_4 = n_5 = n_6$ 0. From (3.2.4), the multiplicities d_λ of the six inequivalent irreducible representations are $d_1 = 1$, $d_2 = 2$, $d_4 = 1$, $d_3 = d_5 = d_6 = 0$. If $P(g)$ is a matrix of the representation $\{P(g)\}$ of PC(2, 2), then

$$P(g) \sim P_1(g) \oplus I_2 \otimes P_2(g) \oplus P_4(g),$$

where $P_1(g)$, $P_2(g)$ and $P_4(g)$ are unitary matrices of degrees one, six and eight, respectively ($n_\lambda \equiv \chi_\lambda$ (identity)). Note that the multiplicities could have also been determined from (3.2.3) by the use of the characters $\chi(C_i)$ ($i = 1, 2, \ldots, 6$) of the permutation representation of PC(2, 2). Table 4 contains these characters, found with the aid of a computer. It is also of interest to note that the partition of the number of flags $21 = 1 + 6 + 6 + 8$ corresponds to the partition of the degrees of freedom in the usual analysis of variance of such a symmetric design: one for the constant, six for the treatments, six for the blocks and eight for error.

The irreducible representations $\{P_1(g)\}$, $\{P_2(g)\}$ and $\{P_4(g)\}$ are all equivalent to sets of real (orthogonal) matrices, i.e., they are all of the first kind. We obtained a count of the number of g^2 in each conjugacy class in order to employ Wigner's test (3.2.5):

$$\varepsilon_1 = 1$$

$$\varepsilon_2 = \frac{1}{168}(22 \times 6 + 42 \times 2 - 24 - 24) = 1$$

$$\varepsilon_4 = \frac{1}{168}(22 \times 8 + 56 - 24 + 24) = 1.$$

The values of one for ε_1, ε_2 and ε_4 imply that the irreducible representations are real.

TABLE 4

Cycles	1^7	$1^3 2^2$	124	13^2	7	7
$\chi(C_i)$	21	5	0	1	0	0
Number of g^2 in each Conjucacy Class	22	42	0	56	24	24

If V is a matrix belonging to the commutant algebra of the group of matrices permuting the flags of the design under the action of PC(2, 2), then by (3.2.1),

$$V \sim a \oplus B_2 \otimes I_6 \oplus cI_8$$

$$= \begin{pmatrix} a & & & \\ & b_{11}I_6 & b_{12}I_6 & \\ & b_{21}I_6 & b_{22}I_6 & \\ & & & cI_8 \end{pmatrix}.$$

The set of real matrices V which are in the commutant algebra will have dimension $\sum_{\lambda=1}^{6} d_\lambda^2 = 6$; the set of real symmetric matrices which are in the commutant algebra will have dimension $\ell = \frac{1}{2}(\sum_{\lambda=1}^{6} d_\lambda^2 + \sum_{\lambda=1}^{6} \varepsilon_\lambda d_\lambda) = 5$

(by (3.2.6)). Section 5.1 of Chapter V contains a discussion of basis matrices for the commutant algebra.

This work may be summarized as:

Theorem 4.1 Let $\{P(g)\}$ be the permutation representation of the collineation group PC(2, 2) of PG(2, 2) (reflecting the permutation of the flags of the design under the action of the group). Then $\{P(g)\}$ is equivalent to the following direct sum of irreducible matrix representations:

$$\{P(g)\} \sim \{P_1(g) \oplus I_2 \otimes P_2(g) \oplus P_4(g)\},$$

where $P_1(g)$, $P_2(g)$ and $P_4(g)$ are real orthogonal matrices of degrees one, six and eight, respectively. The commutant algebra of this representation has dimension six over the reals; the set of real, symmetric matrices in the commutant algebra has dimension five.

4.3 Decomposition of the Permutation Representation of EC(2, 3) into a Direct Sum of Irreducible Representations

The desarguesian finite affine plane EG(2, 3) determines a BIB design with parameters $v = 9$, $b = 12$, $r = 4$, $k = 3$, $\lambda = 1$. We assign to the nine points of this plane the letters A, B, C, D, E, F, G, H and I as follows:

x_1	0	0	0	1	1	1	2	2	2
x_2	0	1	2	0	1	2	0	1	2
	A	B	C	D	E	F	G	H	I

The lines of the plane are indicated by the columns in the following diagram:

$x_1 =$			$x_2 =$			$x_1 + x_2 =$			$x_1 + 2x_2 =$		
0	1	2	0	1	2	0	1	2	0	1	2
A	D	G	A	B	C	A	B	C	A	C	B
B	E	H	D	E	F	F	D	E	E	D	F
C	F	I	G	H	I	H	I	G	I	H	G

Clearly, the design derived from this plane is *resolvable*, i.e., it is possible to separate the twelve blocks into four sets (corresponding to the linear forms x_1, x_2, $x_1 + x_2$ and $x_1 + 2x_2$) such that each treatment occurs exactly once in each set.

The full collineation group EC(2, 3) of the plane EG(2, 3) is determined by the 432 affine transformations

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix} .$$

The matrices $A = ((a_{ij}))$ are the 2×2 non-singular matrices with coefficients in GF(3), while the vectors $\underline{b} = (b_1, b_2)'$ are the 2×1 vectors with coefficients in GF(3). Considered as a permutation group on the 9 points, EC(2, 3) is doubly transitive and has eleven conjugacy classes. Table 5 contains the character table for the eleven irreducible inequivalent representations of EC(2, 3) (Littlewood (1950)).

The group of permutations of these 9 points induces a transitive group on the flags. The group of 36×36 matrices reflecting the flag per-

TABLE 5

Character Table of EC(2, 3), $|EC(2, 3)| = 432$

	cycles (conjugacy classes)	1^9	$1^3 2^3$	$1^3 3^2$	18	18	14^2	126	12^4	3^3	3^3	36
λ	order of i^{th} class γ_i	1	36	24	54	54	54	72	9	8	48	72
1		1	1	1	1	1	1	1	1	1	1	1
2		1	-1	1	-1	-1	1	1	1	1	1	-1
3		2	0	-1	0	0	2	-1	2	2	-1	0
4		2	0	-1	$i\sqrt{2}$	$-i\sqrt{2}$	0	1	-2	2	-1	0
5		2	0	-1	$-i\sqrt{2}$	$i\sqrt{2}$	0	1	-2	2	-1	0
6		3	1	0	-1	-1	-1	0	3	3	0	1
7		3	-1	0	1	1	-1	0	3	3	0	-1
8		8	2	2	0	0	0	0	0	-1	-1	-1
9		8	-2	2	0	0	0	0	0	-1	-1	1
10		16	0	-2	0	0	0	0	0	-2	1	0
11		4	0	1	0	0	0	-1	-4	4	1	0

mutations is a representation of the group of point permutations determined from EC(2, 3); we wish to determine the decomposition of this representation into a direct sum of irreducible representations. Let H be the subgroup of EC(2, 3) whose elements fix the flag consisting of the point (0, 0) and the line $x_2 = 0$. This subgroup H has order $\frac{|EC(2, 3)|}{n} = \frac{432}{36} = 12$ and is induced by the following matrices (which transform the point vectors):

Matrices of H	Point Permutations	Cycles
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	identity	1^9
$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	(BEH) (CIF)	$1^3 3^2$
$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$	(BHE) (CFI)	$1^3 3^2$
$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$	(BC) (EF) (HI)	$1^3 2^3$
$\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$	(BF) (CH) (EI)	$1^3 2^3$
$\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$	(BI) (CE) (FH)	$1^3 2^3$
$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$	(DG) (EH) (FI)	$1^3 2^3$
$\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$	(BE) (CI) (DG)	$1^3 2^3$
$\begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}$	(BH) (CF) (DG)	$1^3 2^3$
$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$	(BC) (DG) (EI) (FH)	12^4

$$\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \qquad \text{(BFECHI) (DG)} \qquad 126$$

$$\begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix} \qquad \text{(BIHCEF) (DG)} \qquad 126$$

These matrices can be determined by the transformations expressed in (3.3.9).

Table 6 contains the values η_i, $i = 1, 2, \ldots, 11$ where η_i is the number of elements of the stabilizer H in the conjugacy class C_i. From (3.2.4), the multiplicities d_λ of the eleven inequivalent irreducible representations are $d_1 = d_6 = d_{10} = 1$, $d_8 = 2$ and $d_2 = d_3 = d_4 = d_5 = d_7 = d_9 = d_{11} = 0$. Thus, for any matrix $P(g)$ in the (permutation) matrix representation $\{P(g)\}$ of $EC(2, 3)$,

$$P(g) \sim P_1(g) \oplus P_6(g) \oplus I_2 \otimes P_8(g) \oplus P_{10}(g),$$

where $P_1(g)$, $P_6(g)$, $P_8(g)$ and $P_{10}(g)$ are unitary matrices of degrees one, three, eight and sixteen, respectively. The d_λ's could also have been determined from $d_\lambda = \frac{1}{432} \sum_{i=1}^{11} \gamma_i \overline{\chi_\lambda}(C_i)\, \chi(C_i)$ (3.2.3). Table 6 contains the values $\chi(C_i)$, found with the aid of a computer. The values γ_i and $\chi_\lambda(C_i)$ are found in the table of characters, Table 5. We note here also that the partition of the number of flags

$$36 = 1 + 3 + 8 + 8 + 16$$

corresponds to the partition of the degrees of freedom in the usual analysis of variance for such a resolvable design: one for the constant, three for the replications, eight for the treatments, eight for the blocks within replications, and sixteen for error.

TABLE 6

Cycles	1^9	$1^3 2^3$	$1^3 3^2$	18	18	14^2	126	12^4	3^3	3^3	36
η_i	1	6	2	0	0	0	2	1	0	0	0
$\chi(C_i)$	36	6	3	0	0	0	2	1	0	0	0
Number of g^2 in each conjugacy class	46	0	36	0	0	108	0	54	80	48	0

The last row of Table 6 contains the number of g^2 in each conjugacy class, $g \in EC(2, 3)$; these g^2 are used in Wigner's test. From (3.2.5),

$$\varepsilon_1 = \frac{1}{432}(46 + 96 + 108 + 54 + 80 + 48) = 1$$

$$\varepsilon_2 = \frac{1}{432}(46 \times 3 - 108 + 54 \times 3 + 80 \times 3) = 1$$

$$\varepsilon_8 = \frac{1}{432}(46 \times 8 + 96 \times 2 - 80 - 48) = 1$$

$$\varepsilon_{16} = \frac{1}{432}(46 \times 4 + 96 - 54 \times 4 + 80 \times 4 + 48) = 1.$$

Since the values for ε_1, ε_6, ε_8 and ε_{10} are all equal to one, the irreducible representations $\{P_1(g)\}$, $\{P_6(g)\}$, $\{P_8(g)\}$ and $\{P_{10}(g)\}$ are all equivalent to sets of real (orthogonal) matrices.

If V is a matrix belonging to the commutant algebra of the permutation representation of EC(2, 3), then by (3.2.1),

$$V \sim a \oplus bI_3 \oplus C_2 \otimes I_8 \oplus dI_{16}$$

$$= \begin{pmatrix} a & & & & \\ & bI_3 & & & \\ & & c_{11}I_8 & c_{12}I_8 & \\ & & c_{21}I_8 & c_{22}I_8 & \\ & & & & dI_{16} \end{pmatrix}.$$

The commutant algebra of real matrices will have dimension $\sum_{\lambda=1}^{11} d_\lambda^{\,2} = 7$; the set of real symmetric matrices in the commutant algebra will have dimension $\ell = \frac{1}{2}(\sum_{\lambda=1}^{11} d_\lambda^{\,2} + \sum_{\lambda=1}^{11} \varepsilon_\lambda d_\lambda) = 6$ (3.2.6). Section 5.3 of Chapter V contains a discussion of the basis matrices for the commutant algebra.

We summarize:

Theorem 4.2 Let $\{P(g)\}$ be the representation of the collineation group EC(2, 3) of EG(2, 3) (reflecting the permutation of the flags of the design under the action of EC(2, 3)). $\{P(g)\}$ is equivalent to a direct sum of irreducible matrix representations such that

$$\{P(g)\} \sim \{P_1(g) \oplus P_6(g) \oplus I_2 \otimes P_8(g) \oplus P_{10}(g)\}$$

where $P_1(g)$, $P_6(g)$, $P_8(g)$ and $P_{10}(g)$ are real orthogonal matrices of degrees one, three, eight and sixteen, respectively. The commutant algebra of the representation has dimension seven over the reals; the set of matrices in this algebra which are real symmetric has dimension six.

V. DETERMINATION OF THE COMMUTANT ALGEBRA FOR BALANCED INCOMPLETE BLOCK DESIGNS BASED ON FINITE GEOMETRIES

5.1 The Design Based on the Desarguesian Finite Projective Plane PG(2, s), $s = p^r$

A finite projective plane defines a symmetric BIB design with parameters $v = b = s^2 + s + 1$, $r = k = s + 1$, $\lambda = 1$ (3.3.1). (The treatments are defined to be points of the plane while the blocks are defined to be the lines.) For any such design, there will be $(s^2 + s + 1)(s + 1)$ flags. The full collineation group PC(2, s) of the desarguesian finite plane PG(2, s) $(s = p^r)$ is defined by the $rs^3(s^2 + s + 1)(s + 1)(s - 1)^2$ point transformations

$$A\underset{\sim}{x}_m = \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_0^m \\ x_1^m \\ x_2^m \end{pmatrix} = \alpha \begin{pmatrix} x_0^* \\ x_1^* \\ x_2^* \end{pmatrix} \quad (\alpha(\neq 0) \in GF(s);\ m = p^u;\ u = 0, 1, \ldots, r-1),$$

where A is non-singular with coefficients in GF(s). The subgroup of collineations H which fix the point (1 0 0) and the line (0 0 1) $(x_2 = 0)$ are from (3.3.5) those of the form

$$(5.1.1)\quad A\underset{\rightarrow}{x}_m = \begin{pmatrix} 1 & a_{01} & a_{02} \\ 0 & a_{11} & a_{12} \\ 0 & 0 & a_{22} \end{pmatrix} \begin{pmatrix} x_0^m \\ x_1^m \\ x_2^m \end{pmatrix} = \alpha \begin{pmatrix} x_0^* \\ x_1^* \\ x_2^* \end{pmatrix}, \ \alpha(\neq 0) \in GF(s).$$

Clearly a_{11} and a_{22} are non-zero. For any such transformation, the lines are transformed as follows (Theorem 3.2):

$$(5.1.2)\quad \begin{pmatrix} 1 & 0 & 0 \\ a_{01} & a_{11} & 0 \\ a_{02} & a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} y_0^* \\ y_1^* \\ y_2^* \end{pmatrix} = \gamma \begin{pmatrix} y_0^m \\ y_1^m \\ y_2^m \end{pmatrix}, \ \gamma(\neq 0) \in GF(s).$$

The stabilizer H will have order $r(s-1)^2 s^3$.

We wish to use Theorem 3.1 to determine the dimension of the commutant algebra corresponding to the full collineation group PC(2, s). We thus need to find the number of orbits of H which, from (3.1.1) is equal to $\frac{1}{|H|} \sum_{g \in H} \Psi(g)$, where $\Psi(g)$ is the number of fixed flags under the action of $g \in PC(2, s)$. To find this number, we consider the conditions under which any given flag is fixed and then count the number of transformations in H satisfying these conditions. Types of lines and types of points will first be considered.

For any $m = p^j$ $(j = 0, 1, \ldots, r - 1)$, we have the following line types:

Line type	Number of lines of this type	Conditions under which a line of this type is fixed (5.1.2)
(0 0 1)	1	fixed for all A in H
(0 1 a); $a \in GF(s)$	s	$a_{12} + a_{22}a = a_{11}a^m$
(1 a b); $a, b \in GF(s)$	s^2	$a_{01} + a_{11}a = a^m$ $a_{02} + a_{12}a + a_{22}b = b^m$
	$s^2 + s + 1$ lines	

The third column contains the conditions under which a line of a certain type is fixed given that the lines are transformed by a matrix in the stabilizer H. For example, a line of the type (0 1 a), $a \in GF(s)$, is fixed if

$$\begin{pmatrix} 1 & 0 & 0 \\ a_{01} & a_{11} & 0 \\ a_{02} & a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ a \end{pmatrix} = \gamma \begin{pmatrix} 0 \\ 1 \\ a^m \end{pmatrix}, \quad \gamma (\neq 0) \in GF(s).$$

We thus have the conditions as stated.

Likewise, for a given value of m, we have the following point types:

Point Type	Number of points of this type	Conditions under which a point of this type is fixed (5.1.1)
(1 0 0)	1	fixed for all A in H
(c 1 0); $c \in GF(s)$	s	$c^m + a_{01} = a_{11}c$
(c d 1); $c, d \in GF(s)$	s^2	$c^m + a_{01}d^m + a_{02} = a_{22}c$ $a_{11}d^m + a_{12} = a_{22}d$
	$s^2 + s + 1$ points	

For each point type, we determine the lines on which it appears (Table 7, Column 3). Then by examining the conditions under which a given flag (point-line pair) is fixed, we determine the number of transformation matrices in the stabilizer H satisfying these conditions. (These counts hold for any fixed $m = p^u$, $u = 0, 1, \ldots, r - 1$.) This will give the number of transformations fixing the flag (5). The counts are found in Appendix B. For each flag type, column (6) contains the total number of transformations fixing the flag type, i.e., the product of the value in (4) and the value in (5). Column (7) contains the total number of transformations fixing a point type (the sum of the values in (6) for a point type multiplied by the value in (2)). The total number of fixed flags under the transformations (5.1.1) (for a fixed m) is the sum of the entries in column (7); the value here is $6(s - 1)^2s^3$. Since there are r values of m, $\sum_{g \in H} \Psi(g) = 6r(s - 1)^2s^3$. The dimension of the commutant

TABLE 7

(for transformations (5.1.1) for a fixed m)

(1) Point type	(2) Number of points of this type	(3) Lines which contain the point	(4) Number of lines of this type which contain the point	(5) Number of collineations fixing a given flag	(6) Number of collineations fixing a flag type	(7) Total number of collineations fixing a point type
(1 0 0)	1	a) (0 0 1) b) (0 1 a), $a \in GF(s)$	1 s	$(s-1)^2s^3$ $(s-1)^2s^2$	$(s-1)^2s^3$ $(s-1)^2s^3$	$2(s-1)^2s^3$
(c 1 0); $c \in GF(s)$	s	a) (0 0 1) b) (1 a b), $a = -c$, $b \in GF(s)$	1 s	$(s-1)^2s^2$ $(s-1)^2s$	$(s-1)^2s^2$ $(s-1)^2s^2$	$2(s-1)^2s^3$
(c d 1); $c, d \in GF(s)$	s^2	a) (0 1 a), $a = -d$ b) (1 a b), $b = -c - ad$, $a \in GF(s)$	1 s	$(s-1)^2s$ $(s-1)^2$	$(s-1)^2s$ $(s-1)^2s$	$2(s-1)^2s^3$
						$6(s-1)^2s^3$

algebra is thus

$$\frac{1}{|H|}\sum_{g\in H}\Psi(g) = \frac{6r(s-1)^2s^3}{r(s-1)^2s^3} = 6.$$

The six dimensional algebra $\mathcal{R}_6$ generated by the relationship matrices I, G, B, T, BT and TB coincides with the commutant algebra of $\{P(g)\}$, the (flag) permutation representation of the collineation group PC(2, s). This is true since we know from Chapter II that $\mathcal{R}_6$ is a six-dimensional subalgebra of the commutant algebra of $\{P(g)\}$ which is seen here to be six dimensional. (Note our design is symmetric.) The matrices I, G, B, T and BT + TB provide a linear basis for the real symmetric matrices in the algebra; these matrices, however, do not generate an algebra (e.g., from Table 1, $T(BT + TB) = \lambda G + (r - \lambda)T + rTB$. Alternatively, we may choose the linear basis for $\mathcal{R}_6$ introduced by Sysoev and Shaikin (1967, II); this basis consists of matrices $E_0, E_1, \ldots, E_5$ as defined in Chapter II (2.1). The matrices $B(\Delta_i) = B_i$ $(i = 1, 2, \ldots, 6)$ which correspond to the six orbits of H, may also be used as a linear basis for $\mathcal{R}_6$.

In summary,

Theorem 5.1 Let $\{P(g)\}$ be the permutation representation of the collineation group PC(2, s) of PG(2, s) ($s = p^r$, p a prime, r an integer) reflecting the permutation of the incident point-line pairs (flags) under the action of the group. The commutant algebra of this representation has dimension six. The matrices I, G, B, T, BT and TB form a linear basis for this algebra; the matrices I, G, B, T and BT + TB form a basis for the linear space of symmetric matrices in this algebra.

5.2 The Design Based on the Desarguesian Finite Projective Geometry PG(k, s), $k \geq 3$, $s = p^r$

We know from (3.3.1) that a finite projective geometry PG(k, s), $k \geq 3$ defines a symmetric design with parameters $v = b = \frac{s^{k+1} - 1}{s - 1}$, $r = k = \frac{s^k - 1}{s - 1}$ and $\lambda = \frac{s^{k-1} - 1}{s - 1}$, if the blocks of the design are hyperplanes of the geometry. The collineations of such a geometry (and hence, the automorphisms of our designs based on this geometry) are given by (3.3.3). The collineations which fix the point (1, 0, ..., 0) and the hyperplane (0, 0, ..., 0, 1) are given by (3.3.6), i.e., the $rs^{2k-1}(s - 1)\prod_{i=0}^{k-2}(s^{k-1} - s^i)$ transformations of the form

$$(5.2.1) \qquad \underline{Ax}_m = \begin{pmatrix} 1 & \underline{a}_1' & a_{0k} \\ \underline{0} & A_1 & \underline{a}_2 \\ 0 & \underline{0}' & a_{kk} \end{pmatrix} \begin{pmatrix} x_0^m \\ x_1^m \\ \vdots \\ x_k^m \end{pmatrix} \quad (m = p^u;\ u = 0, 1, \ldots, r - 1;\ a_{ij} \in GF(s)),$$

where $\underline{a}_1 = (a_{01}\ a_{02} \cdots a_{0,\,k-1})'$, $\underline{a}_2 = (a_{1k}\ a_{2k} \cdots a_{k-1,\,k})'$,

$$A_1 = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1,\,k-1} \\ a_{21} & a_{22} & \cdots & a_{2,\,k-1} \\ & & \vdots & \\ a_{k-1,\,1} & a_{k-2,\,2} & \cdots & a_{k-1,\,k-1} \end{pmatrix}.$$

A is non-singular; hence, A_1 is non-singular and $a_{kk} \neq 0$. Hyperplanes are transformed by the map $(A')^{-1}(y_0^m\ y_1^m \ldots y_k^m)'$.

As in Section 5.1, we wish to determine the dimension of the commutant algebra corresponding to the full collineation group PC(k, s), $k \geq 3$, by counting the number of fixed flags under the action of PC(k, s) and then determining the number of orbits of H from (3.1.1). We thus consider point types and hyperplane types. For any $m = p^u$ $(u = 0, 1, \ldots, r - 1)$, we have the following hyperplane types:

Hyperplane type	Number of hyperplanes of this type	Conditions under which a hyperplane of this type is fixed
$\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$	1	fixed for all A in H
$\begin{pmatrix} 0 \\ b_1 \\ b_2 \\ \vdots \\ b_k \end{pmatrix}$ $b_i \in GF(s)$; $b_1, b_2, \ldots, b_{k-1}$ not all zero	$\frac{s(s^{k-1} - 1)}{s - 1}$	$A_1' \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{k-1} \end{pmatrix} = \gamma \begin{pmatrix} b_1^m \\ b_2^m \\ \vdots \\ b_{k-1}^m \end{pmatrix}$, $\gamma(\neq 0) \in GF(s)$ $a_{1k}b_1 + a_{2k}b_2 + \ldots + a_{kk}b_k = \gamma b_k^m$

Hyperplane type	Number of hyperplanes of this type	Conditions under which a hyperplane of this type is fixed
$\begin{pmatrix} 1 \\ b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_k \end{pmatrix}$ $b_i \in GF(s)$	s^k	$\begin{pmatrix} a_{01} \\ a_{02} \\ \cdot \\ \cdot \\ \cdot \\ a_{0,\,k-1} \end{pmatrix} + A_1' \begin{pmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_{k-1} \end{pmatrix} = \begin{pmatrix} b_1^m \\ b_2^m \\ \cdot \\ \cdot \\ \cdot \\ b_{k-1}^m \end{pmatrix}$ $a_{0k} + a_{1k}b_1 + a_{2k}b_2 + \ldots + a_{kk}b_k = b_k^m$
	$\frac{s^{k+1} - 1}{s - 1}$ hyperplanes	

Likewise, for any given value of m, we have the following point types:

Point type	Number of points of this type	Conditions under which a point of this type is fixed (5.2.1)
$\begin{pmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix}$	1	fixed for all A in H

Point type	Number of points of this type	Conditions under which a point of this type is fixed (5.2.1)
$\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \\ 0 \end{pmatrix}$ $c_i \in GF(s)$; $c_2, c_3, \ldots, c_k$ not all zero	$\frac{s(s^{k-1} - 1)}{s - 1}$	$c_1^m + a_{01}c_2^m + a_{02}c_3^m + \ldots + a_{0,k-1}c_k^m = \alpha c_1$ $A_1 \begin{pmatrix} c_2^m \\ c_3^m \\ \vdots \\ c_k^m \end{pmatrix} = \alpha \begin{pmatrix} c_2 \\ c_3 \\ \vdots \\ c_k \end{pmatrix}$, $\alpha(\neq 0) \in GF(s)$
$\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \\ 1 \end{pmatrix}$ $c_i \in GF(s)$	s^k	$c_1^m + a_{01}c_2^m + a_{02}c_3^m + \ldots + a_{0,k-1}c_k^m + a_{0k} = a_{kk}c$ $A_1 \begin{pmatrix} c_2^m \\ c_3^m \\ \vdots \\ c_k^m \end{pmatrix} + \begin{pmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a_{k-1,k} \end{pmatrix} = a_{kk} \begin{pmatrix} c_2 \\ c_3 \\ \vdots \\ c_k \end{pmatrix}$
	$\frac{s^{k+1} - 1}{s - 1}$ points	

In Table 8, for each point type, we determine the hyperplanes on which it appears. We examine the conditions under which a given flag (incident point-hyperplane) is fixed and hence find the number of transformations fixing a flag (5). Note that we used the notation $\Phi(s, q)$ to

TABLE 8

(for transformations (5.2.1) for a fixed m, $\Phi(s, q) = \prod_{i=0}^{q-1} (s^q - s^i)$)

(1) Point type	(2) Number of points of this type	(3) Hyperplanes which contain the point	(4) Number of hyperplanes of this type which contain the point	(5) Number of collineations fixing a given flag	(6) Number of collineations fixing a flag type	(7) Total number of collineations fixing a point type
$(1, 0, \ldots, 0)$	1	a) $(0, 0, \ldots, 0, 1)$	1	$(s-1)s^{2k-1}\Phi(s, k-1)$	$(s-1)s^{2k-1}\Phi(s, k-1)$	$2(s-1)s^{2k-1}\Phi(s, k-1)$
		b) $(0, b_1, b_2, \ldots, b_k)$; $b_i \in GF(s)$; $b_1, b_2, \ldots, b_{k-1}$ not all zero	$\frac{s^k - s}{s - 1}$	$(s-1)^2 s^{3k-4}\Phi(s, k-2)$	$(s-1)s^{2k-1}\Phi(s, k-1)$	
$(c_1, c_2, \ldots, c_k, 0)$; $c_i \in GF(s)$; $c_2, c_3, \ldots, c_k$ not all zero	$\frac{s^k - s}{s - 1}$	a) $(0, 0, \ldots, 0, 1)$	1	$(s-1)^2 s^{3k-4}\Phi(s, k-2)$	$(s-1)^2 s^{3k-4}\Phi(s, k-2)$	$3(s-1)s^{2k-1}\Phi(s, k-1)$
		b) $(0, b_1, b_2, \ldots, b_k)$; $b_i \in GF(s)$; $b_1, b_2, \ldots, b_{k-1}$ not all zero; $c_2b_1+c_3b_2+\ldots+c_kb_{k-1}=0$	$\frac{s^{k-1} - s}{s - 1}$	$(s-1)^3 s^{4k-8}\Phi(s, k-3)$, $k \geq 4$; $(s-1)^3 s^4$, $k = 3$	$(s-1)^2 s^{3k-4}\Phi(s, k-2)$	
		c) $(1, b_1, b_2, \ldots, b_k)$; $b_i \in GF(s)$; $c_1+c_2b_1+\ldots+c_kb_{k-1}=0$	$s^k - 1$	$(s-1)^2 s^{2k-3}\Phi(s, k-2)$	$(s-1)^2 s^{3k-4}\Phi(s, k-2)$	
$(c_1, c_2, \ldots, c_k, 1)$, $c_i \in GF(s)$	s^k	a) $(0, b_1, b_2, \ldots, b_k)$; $b_i \in GF(s)$; $b_1, b_2, \ldots, b_{k-1}$ not all zero; $c_2b_1+c_3b_2+\ldots+c_kb_{k-1}+b_k=0$	$\frac{s^{k-1} - 1}{s - 1}$	$(s-1)^2 s^{2k-3}\Phi(s, k-2)$	$(s-1)s^{k-1}\Phi(s, k-1)$	$2(s-1)s^{2k-1}\Phi(s, k-1)$
		b) $(1, b_1, b_2, \ldots, b_k)$; $b_i \in GF(s)$; $c_1+c_2b_1+c_3b_2+\ldots+c_kb_{k-1}+b_k=0$	s^{k-1}	$(s-1)\Phi(s, k-1)$	$(s-1)s^{k-1}\Phi(s, k-1)$	
						$7(s-1)s^{2k-1}\Phi(s, k-1)$ $=7(s-1)s^{2k-1}\prod_{i=2}^{k-1}(s^{k-1}-s^i)$

denote $\prod_{i=0}^{q-1} (s^q - s^i)$. (The counts are found in Appendix C.) Columns (6) and (7) are found as for the case PG(2, s). The total number of fixed flags under the transformations (5.2.1) is found to be $7(s-1)s^{2k-1} \prod_{i=0}^{k-2} (s^{k-1} - s^i)$ for a fixed m. Since there are r values of m, $\sum_{g\in H} \Psi(g) = 7r(s-1)s^{2k-1} \prod_{i=0}^{k-2} (s^{k-1} - s^i)$. The dimension for the commutant algebra of the matrix representation $\{P(g)\}$ is therefore

$$\frac{1}{|H|} \sum_{g\in H} \Psi(g) = \frac{7r(s-1)s^{2k-1} \prod_{i=0}^{k-2} (s^{k-1} - s^i)}{r(s-1)s^{2k-1} \prod_{i=0}^{k-2} (s^{k-1} - s^i)} = 7.$$

The matrices I, G, B, T, BT, TB and S (introduced in Chapter II) serve as a linear basis for the real matrices in the commutant algebra, i.e., the commutant algebra over R is the extended relationship algebra P_7. S is linearly independent of the other matrices if $\lambda > 1$, which is true for $k \geq 3$. Also, since the design is symmetric, the matrix $BTB = (r - \lambda)B + \lambda G$.

Theorem 5.2 Let $\{P(g)\}$ be the group of matrices which permute the incident point-hyperplane pairs (flags) of PG(k, 3), $k \geq 3$, under the action of the group PC(k, s) ($s = p^r$, p a prime, r an integer). The commutant algebra for this matrix has dimension seven. The matrices I, G, B, T, BT, TB and S form a basis for the commutant algebra; the matrices I, G, B, T, BT + TB and S form a basis for the linear space of real symmetric matrices in this algebra.

5.3 The Design Based on the Desarguesian Finite Affine Plane EG(2, s), $s = p^r$

A finite affine plane defines an asymmetric BIB design with parameters $v = s^2$, $b = s^2 + s$, $r = s + 1$, $k = s$, $\lambda = 1$ (3.3.2) and

$s^2(s + 1)$ flags. The full collineation group EC(2, s) of the finite affine plane EG(2, s) is defined by the $r(s - 1)^2(s + 1)s^3$ treatment transformations

$$A\underset{\sim}{x}_m + \underset{\sim}{c} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1^m \\ x_2^m \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix} \quad (a_{ij},\ c_i \in GF(s);\ m = p^u;\ u = 0, 1, \ldots, r-1),$$

such that A is non-singular. From (3.3.9), the collineations

$$(5.3.1) \qquad A\underset{\sim}{x}_m = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} \begin{pmatrix} x_1^m \\ x_2^m \end{pmatrix} = \begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix}$$

define the subgroup H of EC(2, s) which fixes the point (0 0) and the line (0 0 1) ($x_2 = 0$). Clearly a_{11} and a_{22} in (5.3.1) are non-zero. The line (y_0, y_1, y_2) is transformed as follows (Theorem 3.3):

$$(5.3.2) \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_{11} & 0 \\ 0 & a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} y_0^* \\ y_1^* \\ y_2^* \end{pmatrix} = \gamma \begin{pmatrix} y_0^m \\ y_1^m \\ y_2^m \end{pmatrix}, \quad \gamma (\neq 0) \in GF(s).$$

The stabilizer H consists of $r(s - 1)^2 s$ elements.

Proceeding as in Sections 5.1 and 5.2, for any $m = p^u$, $u = 0, 1, \ldots, r - 1$, we consider the following line types:

Line type	Number of lines of this type	Conditions under which a line of this type is fixed (5.3.2)
(0 0 1)	1	fixed for all A in H
(0 1 a); $a \in GF(s)$	s	$a_{12} + a_{22}a = a_{11}a^m$

Line type	Number of lines of this type	Conditions under which a line of this type is fixed (5.3.2)
(1 0 a); $a \in GF(s)$	$s - 1$	$a_{22}a = a^m$
(1 a b); $a, b \in GF(s)$ $a \neq 0$	$s(s - 1)$	$a_{11}a = a^m$ $a_{12}a + a_{22}b = b^m$
	$s^2 + s$ lines	

Likewise, for any given value of m, we have the following point types:

Point type	Number of points of this type	Conditions under which a point of this type is fixed (5.3.1)
(0 0)	1	fixed for all A in H
(c 0); $c(\neq 0) \in GF(s)$	$s - 1$	$a_{11}c^m = c$
(c d); $d \neq 0$; $c, d \in GF(s)$	$s(s - 1)$	$a_{11}c^m + a_{12}d^m = c$ $a_{22}d^m = d$
	s^2 points	

For each flag, the number of transformations which fix that flag is found in Table 9. (The counts are found in Appendix D.) The total number of fixed flags under the transformations (5.3.1) (for a fixed m) is $7s(s - 1)^2$. Since there are r values of m, $\sum_{g \in H} \Psi(g) = 7rs(s - 1)^2$. The dimension of the commutant algebra of the matrix representation $\{P(g)\}$ is therefore

$$\frac{1}{|H|} \sum_{g \in H} \Psi(g) = \frac{7rs(s-1)^2}{rs(s-1)^2} = 7.$$

TABLE 9

(for transformations (5.3.1) for a fixed m)

(1) Point type	(2) Number of points of this type	(3) Lines which contain the point	(4) Number of lines of this type which contain the point	(5) Number of collineations fixing a given flag	(6) Number of collineations fixing a flag type	(7) Total number of collineations fixing a point type
(0 0)	1	a) (0 0 1)	1	$(s-1)^2 s$	$(s-1)^2 s$	$2(s-1)^2 s$
		b) (0 1 a), $a \in GF(s)$	s	$(s-1)^2$	$(s-1)^2 s$	
(c 0); $c(\neq 0) \in GF(s)$	$s-1$	a) (0 0 1)	1	$(s-1)s$	$(s-1)s$	$2(s-1)^2 s$
		b) (1 a b), $a = -c^{-1}$, $b \in GF(s)$	s	$s-1$	$(s-1)s$	
(c d); $c, d \in GF(s)$; $d \neq 0$	$(s-1)s$	a) (0 1 a), $a = -cd^{-1}$	1	$s-1$	$s-1$	$3(s-1)^2 s$
		b) (1 0 a), $a = -d^{-1}$	1	$s-1$	$s-1$	
		c) (1 a b), $a(\neq 0) \in GF(s)$, $b = -(1+ac)d^{-1}$	$s-1$	1	$s-1$	
						$7(s-1)^2 s$

Since we have a symmetric design with $\lambda = 1$, the matrices I, G, B, T, BT, TB and BTB serve as a linear basis for the real matrices in the commutant algebra of $\{P(g)\}$, i.e., the commutant algebra over $\mathbb{R}$ is James' relationship algebra R_7.

We therefore have,

Theorem 5.3 Let $\{P(g)\}$ be the permutation matrix representation of the transitive collineation group EC(2, s) of EG(2, s) ($s = p^r$, p a prime, r an integer) reflecting the permutations of the incident point-line pairs (flags) under the action of the group. The dimension of the commutant algebra of this representation is seven. The matrices I, G, B, T, BT, TB and BTB form a linear basis over $\mathbb{R}$ for the commutant algebra; I, G, B, T, BT + TB and BTB form a basis for the linear space of symmetric matrices in this algebra.

5.4 The Design Based on the Desarguesian Finite Affine Geometry EG(k, s), $k \geq 3$, $s = p^r$

From (3.3.2), a finite affine geometry EG(k, s), $k \geq 3$ defines an asymmetric BIB design with parameters $v = s^k$, $b = \frac{s(s^k - 1)}{s - 1}$, $r = \frac{s^k - 1}{s - 1}$, $k = s^{k-1}$ and $\lambda = \frac{s^{k-1} - 1}{s - 1}$ and with $\frac{s^k(s^k - 1)}{s - 1}$ flags. (Flags are taken here to be incident point-hyperplane pairs.) The collineations of such a geometry are given by the treatment transformations (3.3.7). The subgroup H of collineations which fix the point $(0, 0, \ldots, 0)$ and the line $(0, 0, \ldots, 0, 1)$ is determined from (3.3.9), i.e., the $r(s - 1)s^{k-1} \prod_{i=0}^{k-2} (s^{k-1} - s^i)$ transformations of the form

$$(5.4.1)\qquad \underline{Ax}^m = \begin{pmatrix} A_1 & \underline{a} \\ \underline{0}' & a_{kk} \end{pmatrix} \begin{pmatrix} x_1^m \\ x_2^m \\ \vdots \\ x_k^m \end{pmatrix} = \begin{pmatrix} x_1^* \\ x_2^* \\ \vdots \\ x_k^* \end{pmatrix} \quad (a_{ij} \in GF(s);\ m = p^u;\ u = 0, 1, \ldots, r-1),$$

in which

$$A_1 = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1,k-1} \\ a_{21} & a_{22} & \cdots & a_{2,k-1} \\ & & \vdots & \\ a_{k-1,1} & a_{k-1,2} & \cdots & a_{k-1,k-1} \end{pmatrix}, \quad |A_1| \neq 0,$$

$$\underline{a}' = (a_{1k}\ \ a_{2k}\ \cdots\ a_{k-1,k}),$$

$$a_{kk} \neq 0,\ a_{ij} \in GF(s).$$

The hyperplanes are transformed as follows (Theorem 3.3):

$$(5.4.2)\qquad \begin{pmatrix} 1 & \underline{0}' & 0 \\ \underline{0} & A_1' & \underline{0} \\ \underline{0} & \underline{a}' & a_{kk} \end{pmatrix} \begin{pmatrix} y_0^* \\ y_1^* \\ \vdots \\ y_k^* \end{pmatrix} = \gamma \begin{pmatrix} y_0^m \\ y_1^m \\ \vdots \\ y_k^m \end{pmatrix}, \quad \gamma(\neq 0) \in GF(s).$$

As in the previous sections of this chapter, we consider the following hyperplane types (for any $m = p^u$, $u = 0, 1, \ldots, r-1$):

Hyperplane type	Number of hyperplanes of this type	Conditions under which a hyperplane of this type is fixed (5.4.2)
$\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$	1	fixed for all A in H
$\begin{pmatrix} 0 \\ b_1 \\ b_2 \\ \vdots \\ b_{k-1} \\ b_k \end{pmatrix}$ $b_i \in GF(s)$; $b_1, b_2, \ldots, b_{k-1}$ not all zero	$\frac{s(s^{k-1} - 1)}{s - 1}$	$A_1' \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{k-1} \end{pmatrix} = \gamma \begin{pmatrix} b_1^m \\ b_2^m \\ \vdots \\ b_{k-1}^m \end{pmatrix}$, $\gamma (\neq 0) \in GF(s)$ $a_{1k}b_1 + a_{2k}b_2 + \ldots, + + a_{k-1,\,k}b_{k-1} + a_{kk}b_k = \gamma b_k^m$
$\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ b \end{pmatrix}$ $b (\neq 0) \in GF(s)$	$s - 1$	$\gamma = 1$ $a_{kk} = b^{m-1}$

Hyperplane type	Number of hyperplanes of this type	Conditions under which a hyperplane of this type is fixed (5.4.2)
$\begin{pmatrix} 1 \\ b_1 \\ b_2 \\ \vdots \\ b_{k-1} \\ b_k \end{pmatrix}$ $b_i \in GF(s)$; $b_1, b_2, \ldots, b_{k-1}$ not all zero	$s(s^{k-1} - 1)$	$\gamma = 1$ $A_1' \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{k-1} \end{pmatrix} = \begin{pmatrix} b_1^m \\ b_2^m \\ \vdots \\ b_{k-1}^m \end{pmatrix}$ $a_{1k}b_1 + a_{2k}b_2 + \ldots + a_{kk}b_k = b_k^m$
	$\frac{s(s^k - 1)}{s - 1}$ hyperplanes	

Likewise, for any given value of m, we have the following point types:

Point type	Number of points of this type	Conditions under which a point of this type is fixed (5.4.1)
$\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$	1	fixed for all A in H

Point type	Number of points of this type	Conditions under which a point of this type is fixed (5.4.1)
$\begin{pmatrix} c_1 \\ c_2 \\ \cdot \\ \cdot \\ \cdot \\ c_{k-1} \\ 0 \end{pmatrix}$ $c_i \in GF(s)$; $c_1, c_2, \cdots, c_{k-1}$ not all zero	$s^{k-1} - 1$	$A_1 \begin{pmatrix} c_1^m \\ c_2^m \\ \cdot \\ \cdot \\ \cdot \\ c_{k-1}^m \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ \cdot \\ \cdot \\ \cdot \\ c_{k-1} \end{pmatrix}$
$\begin{pmatrix} c_1 \\ c_2 \\ \cdot \\ \cdot \\ \cdot \\ c_{k-1} \\ c_k \end{pmatrix}$ $c_i \in GF(s)$; $c_k \neq 0$	$s^{k-1}(s-1)$	$A_1 \begin{pmatrix} c_1^m \\ c_2^m \\ \cdot \\ \cdot \\ \cdot \\ c_{k-1}^m \end{pmatrix} + c_k^m \begin{pmatrix} a_{1k} \\ a_{2k} \\ \cdot \\ \cdot \\ \cdot \\ a_{k-1,\,k} \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ \cdot \\ \cdot \\ \cdot \\ c_{k-1} \end{pmatrix}$ $a_{kk} = c_k^{1-m}$
	s^k points	

Table 10 contains the number of transformations, for a fixed value of m, which fix any given flag. For such a value of m, the total number of fixed flags under the transformations (5.4.1) is

Table 10

(for transformations (5.4.1), $\Phi(s, q) = \prod_{i=0}^{q-1} (s^q - s^i)$)

(1) Point type	(2) Number of points of this type	(3) Hyperplanes which contain the point	(4) Number of hyperplanes of this type which contain the point	(5) Number of collineations fixing a given flag	(6) Number of collineations fixing a flag type pair	(7) Total number of collineations fixing a point type
$\begin{pmatrix}0\\0\\\cdot\\\cdot\\\cdot\\0\end{pmatrix}$	1	a) $(0, 0, \ldots, 0, 1)$	1	$(s-1)s^{k-1}\Phi(s, k-1)$	$(s-1)s^{k-1}\Phi(s, k-1)$	$2(s-1)s^{k-1}\Phi(s, k-1)$
		b) $(0, b_1, b_2, \ldots, b_k)$; $b_i \in GF(s)$; $b_1, b_2, \ldots, b_{k-1}$ not all zero	$\frac{s(s^{k-1}-1)}{s-1}$	$(s-1)^2 s^{2k-4}\Phi(s, k-2)$	$(s-1)s^{k-1}\Phi(s, k-1)$	
$\begin{pmatrix}c_1\\c_2\\\cdot\\\cdot\\\cdot\\c_{k-1}\\0\end{pmatrix}$ $c_i \in GF(s)$; $c_1, c_2, \ldots, c_{k-1}$ not all zero	$s^{k-1} - 1$	a) $(0, 0, \ldots, 0, 1)$	1	$(s-1)s^{2k-3}\Phi(s, k-2)$	$(s-1)s^{2k-3}\Phi(s, k-2)$	$3(s-1)s^{k-1}\Phi(s, k-1)$
		b) $(0, b_1, b_2, \ldots, b_k)$; $b_i \in GF(s)$; $b_1, b_2, \ldots, b_{k-1}$ not all zero; $c_1b_1+c_2b_2+\ldots+c_{k-1}b_{k-1} = 0$	$\frac{s(s^{k-2}-1)}{s-1}$	$(s-1)^2 s^{3k-7}\Phi(s, k-3)$ for $k \geq 4$, $(s-1)^2 s^2$ for $k = 3$	$(s-1)s^{2k-3}\Phi(s, k-2)$	
		c) $(1, b_1, b_2, \ldots, b_k)$; $b_i \in GF(s)$; $b_1, b_2, \ldots, b_{k-1}$ not all zero; $1+c_1b_1+c_2b_2+\ldots+c_{k-1}b_{k-1} = 0$	s^{k-1}	$(s-1)s^{k-2}\Phi(s, k-2)$	$(s-1)s^{2k-3}\Phi(s, k-2)$	

Table 10
(continued)

(for transformations (5.4.1), $\Phi(s, q) = \prod_{i=0}^{q-1} (s^q - s^i)$)

(1) Point type	(2) Number of points of this type	(3) Hyperplanes which contain the point	(4) Number of hyperplanes of this type which contain the point	(5) Number of collineations fixing a given flag	(6) Number of collineations fixing a flag type pair	(7) Total number of collineations fixing a point type
$\begin{pmatrix} c_1 \\ c_2 \\ \cdot \\ \cdot \\ \cdot \\ c_k \end{pmatrix}$	$(s-1)s^{k-1}$	a) $(0, b_1, b_2, \ldots, b_k)$; $b_i \in GF(s)$; $b_1, b_2, \ldots, b_{k-1}$ not all zero; $c_1b_1+c_2b_2+ \ldots + c_kb_k = 0$	$\frac{(s^{k-1} - 1)}{s - 1}$	$(s-1)s^{k-2}\Phi(s, k-2)$	$\Phi(s, k-1)$	$3(s-1)s^{k-1}\Phi(s, k-1)$
$c_i \in GF(s)$; $c_k \neq 0$		b) $(1, 0, 0, \ldots, 0, b)$; $b = -c_k^{-1}$	1	$\Phi(s, k-1)$	$\Phi(s, k-1)$	
		c) $(1, b_1, b_2, \ldots, b_k)$; $b_i \in GF(s)$; $b_1, b_2, \ldots, b_{k-1}$, not all zero; $1+c_1b_1+c_2b_2+ \ldots + +c_kb_k = 0$	$s^{k-1} - 1$	$s^{k-2}\Phi(s, k-2)$	$\Phi(s, k-1)$	
						$8(s-1)s^{k-1}\Phi(s, k-1)$

$8(s-1)s^{k-1}\prod_{i=0}^{k-2}(s^{k-1}-s^{i})$; thus,

$$\sum_{g\in H}\Psi(g) = 8r(s-1)s^{k-1}\prod_{i=0}^{k-2}(s^{k-1}-s^{i}).$$

For the representation $\{P(g)\}$, the dimension of the commutant algebra is

$$\frac{1}{|H|}\sum_{g\in H}\Psi(g) = \frac{8r(s-1)s^{k-1}\prod_{i=0}^{k-2}(s^{k-1}-s^{i})}{r(s-1)s^{k-1}\prod_{i=0}^{k-2}(s^{k-1}-s^{i})} = 8.$$

(The counts from column (5) are found in Appendix E.) Since we have an asymmetric design with $\lambda > 1$ and dimension eight, the matrices I, G, B, T, BT, TB, BTB and S serve as a linear basis for the real matrices in the commutant algebra.

We summarize in the following theorem:

Theorem 5.4 Let $\{P(g)\}$ be the permutation matrix representation of the transitive collineation group EC(k, s) of EG(k, s) ($k \geq 3$, $s = p^r$, p a prime, r an integer) reflecting the permutations of the incident point-hyperplane pairs (flags) under the action of the group. The commutant algebra of this representation has dimension eight. The matrices I, G, B, T, BT, TB, BTB and S form a linear basis for the algebra of real matrices in this algebra; I, G, B, T, BT + TB, BTB and S form a basis for the linear space of symmetric matrices in this algebra.

VI. ESTIMATION AND ANALYSIS

6.1 Estimation Under the Assumption of a Zero Mean Vector

As mentioned in Chapter II, for a BIB design, Sysoev and Shaikin (1976, III) and Sysoev (1977) determined unbiased estimates for the covariance matrix coefficients c_i under the assumption $\underline{Y} \sim N(0, \sum_{i=1}^{t} c_i V_i)$ where $\sum_{i=1}^{t} c_i V_i$ is a positive definite symmetric matrix in either R_6, R_7, R_7 or R_8. Alternatively, we may use Theorem 1 of Pukelsheim and Styan (1978). Theorem 1 states that if $\underline{Y}_\mu \sim (0, \sum_{i=1}^{t} c_i V_i)$ and the V_i span a k-dimensional special Jordan algebra, then

(i) If $G(\neq 0) \subseteq \mathbb{R}^t$ is the region of values $\underline{c}$ such that $\sum_{i=1}^{t} c_i V_i$ is positive definite, the maximum likelihood estimator for $\underline{c} \in G$ is almost surely equal to the UMVUE (uniformly minimum variance unbiased estimator) $\hat{\underline{c}} = (W'W)^{-1} W'$ vec $\hat{\Sigma}$ where
$W = [\text{vec } V_1 \mid \text{vec } V_2 \mid \ldots \mid \text{vec } V_t]$ and $\hat{\Sigma} = \sum_{\mu=1}^{N} \frac{\underline{Y}_\mu \underline{Y}_\mu'}{N}$.
($N \geq t$; $\underline{Y}_\mu$, $\mu = 1, 2, \ldots, N$ are independent.)

(ii) $\sum_{i=1}^{t} \hat{c}_i V_i$ is a non-negative definite and positive definite if $\hat{\Sigma}$ is positive definite.

(vec V maps a matrix into a vector by ordering the elements lexicographically.)

We first define a *quadratic subspace* of a vector space of real

symmetric matrices A; $\mathcal{D}$ is a quadratic subspace if $D \in \mathcal{D}$ implies $D^2 \in \mathcal{D}$. (This property makes $\mathcal{D}$ a a special *Jordan Algebra*.) From Seely (1971), $\mathcal{D}$ is a quadratic subspace if $(D_i + D_j)^2 \in \mathcal{D}$ for any basis matrices D_i, D_j of $\mathcal{D}$. This will be true for the subspace generated by the matrices I, G, B, T, BT + TB, BTB and S. (If $\lambda > 1$, we assume that we have a design satisfying the block intersection properties given in Chapter II. See Sysoev (1977) for relevant details.)

Now we will consider only the case where $\sum_{i=1}^{t} c_i V_i \in R_7$; the other cases follow with minor modifications. We use the alternative basis $H_1 = E_0 + E_1$, $H_2 = E_2$, ..., $H_6 = E_6$ from (2.1); thus, $W = [\text{vec } H_1 | \text{ vec } H_2 | \ \ldots \ | \text{ vec } H_6]$. Note that $(\text{vec } H_i)' (\text{vec } H_j) = \text{tr } H_i H_j$. Thus, the UMVUE equations reduce to $\sum_{j=1}^{6} \hat{c}_i \text{ tr } (H_i H_j) = \text{tr } (H_i \hat{\Sigma})$, i = 1, 2, ..., 6. These equations were solved in Sysoev (1977); a solution is given in (2.2).

6.2 Estimation Under a Fixed Effects Model

In this section, we assume a fixed effects model, i.e., $E(\underline{Y}) = \mu \underline{j}_n' + D'\tau + L'\underline{\beta}$ under the assumption that our covariance matrix belongs to either R_6, R_7, P_7 or P_8. (Once again, for $\lambda > 1$, we assume our design possesses the block intersection properties expressed in Chapter II.) We first deal here with the notion of complete sufficient statistics for a family of multivariate normal distributions. Seely (1971) assumed a model of the form $\underline{Y} \sim N_n(X\underline{\beta}, \sum_{i=1}^{t} c_i V_i)$ such that $\underline{\beta}$ ranges over $\Omega_1 = \mathbb{R}^p$, $\underline{c} = (c_1, c_2, \ldots, c_t)'$ ranges over Ω_2, a subset of $\mathbb{R}^t$ and each V_i is a known real symmetric matrix such that $V_1 = I$. Additionally, Ω_2 has a non-void open set in $\mathbb{R}^t$, $\sum_{i=1}^{t} c_i V_i$ is positive definite for all $\underline{c} \in \Omega_2$ and $\underline{c}$ and $\underline{\beta}$ are functionally independent. Any

of our four models satisfy these conditions; we know when a covariance matrix in R_6, R_7, P_7 or P_8 is positive definite (Sysoev and Shaikin (1976, I, II) and Sysoev (1977)). Theorem 2 in Seely states that if sp $\{V_1, V_2, \ldots, V_t\}$ is a quadratic subspace of the space A of real symmetric matrices and $V_i R(X) \subseteq R(X)$ for i = 1, 2, ..., t, then the vector statistics $(\underline{Y}'V_1\underline{Y}, \underline{Y}'V_2\underline{Y}, \ldots, \underline{Y}'V_t\underline{Y}, X'\underline{Y})$ are jointly a complete sufficient statistic for our parameters. In fact, if $X = (\underline{j}_n | D' | L')$, then

$$
\begin{aligned}
&IR(X) = R(X)\\
&GR(X) = j_n j_n' R(X) \subseteq R(X)\\
&TR(X) = D'D R(X) \subseteq R(X)\\
&BR(X) = L'L R(X) \subseteq R(X)\\
&(BT + TB)R(X) \subseteq R(X)\\
&BTBR(X) \subseteq R(X)\\
&SR(X) = SR(XX') = SR(G + T + B) = R(\lambda k + r - \lambda)\underline{j}_n'\underline{j}_n +\\
&\quad + (r - \lambda)D'D + \lambda L'LT + (k - \mu)L'L + \mu D'DB) \subseteq R(X).
\end{aligned}
\tag{6.2.1}
$$

(For the last statement, see Sysoev (1977).) Thus since sp {I, G, B, T, BT + TB, BTB, S} is a quadratic subspace of A, Theorem 2 holds.

Furthermore, Seely determined that if Theorem 2 holds and P is the symmetric idempotent matrix such that $R(P) = R(X)$ and $N = I - P$, then the parametric function $\sum_{i=1}^{t} \nu_i c_i$ is estimable in the space $\{\underline{Y}'A\underline{Y} : A \in \lambda\}$ if and only if there exists an $\underline{\alpha}$ such that

$$
\begin{aligned}
&\sum_{i=1}^{t} \operatorname{tr}(V_1 N V_i)\ \alpha_i = \nu_1\\
&\sum_{i=1}^{t} \operatorname{tr}(V_2 N V_i)\ \alpha_i = \nu_2\\
&\qquad\vdots\\
&\sum_{i=1}^{t} \operatorname{tr}(V_t N V_i)\ \alpha_i = \nu_t
\end{aligned}
$$

N in this case is the matrix $E_5(2.1)$. We also have

$$E_5 I = E_5\left[-\frac{1}{\phi}(E_0 + E_1) + \frac{(\phi + 1)}{\phi}(E_2 + E_3) + E_4 + E_5 + E_6\right] = E_5$$

$$E_5 G = E_5[bkE_4] = 0$$

$$E_5 B = E_5[k(E_2 + E_4 + E_6)] = 0$$

$$E_5 T = E_5[r(E_3 + E_4)] = 0$$

$$E_5(BT + TB) = E_5[(r - \lambda)(E_0 + E_1) + 2krE_4] = 0$$

$$E_5 BTB = E_5[k(r - \lambda)E_2 = k^2 rE_4] = 0.$$

Thus if $\lambda = 1$ (and hence S=B+T-I), the only estimable function in the space $\{\underline{Y}'A\underline{Y}: A \in \mathcal{A}\}$ is the function $\nu_1 c_1$ where $\nu_1 c_1$ is the coefficient of the identity. In this case, we have that $\nu_1 c_1$ is estimable if and only if there is an α_1 such that $tr(E_5)\alpha_1 = \nu_1$, i.e., $\alpha_1 = \frac{\nu_1}{n-b-k+1}$. An unbiased estimator for $\nu_1 c_1$ is $\frac{\nu_1}{n-b-k+1}(\underline{Y}'E_5\underline{Y})$; since it is a function of the complete sufficient statistic, it is uniformly minimum variance unbiased.

Concerning estimation of the treatment (and perhaps, block) parameters, we do know that $\underline{a}'\underline{y}$ is a b.l.u.e. for its expectation if and only if $V_i\underline{a} \in \mathcal{R}(X)$ for $i = 1, 2, \ldots, t$. (This is from a result in Seely (1969) which is an extension of a result in Zyskind (1967).) $I\underline{a} \in \mathcal{R}(X)$ implies $\underline{a} \in \mathcal{R}(X)$; from (6.1.1), for any $\underline{a} \in \mathcal{R}(X)$, we also see that $V_i\underline{a} \in \mathcal{R}(X)$. Thus, the coefficient vectors $\underline{a}$ are those vectors and only those vectors which are in $\mathcal{R}(X)$.

6.3 Estimation Under a Random Blocks Model

We now assume a random blocks model, i.e., the model $E(\underline{Y}) = \mu\underline{j}_n + D'\underline{\tau}$, where the covariance matrix of $\underline{Y}$ is contained in one of the algebras R_6, R_7, P_7 or P_8. In the traditional model, $V(\underline{Y}) = \sigma_1^2 L'L + \sigma^2 I = \sigma_1^2 B + \sigma^2 I$; here B and I are in each of the algebras. Also, $R(X) = R(\underline{j}_n | D')$ so that $V_i R(X)$ is not necessarily a subspace of $R(X)$. However, if we assume $\underline{Y} \sim N_n(\mu\underline{j}_n + D'\underline{\tau}, \sum_{i=1}^{t} c_i V_i)$ and $\overline{N} = \{\underline{Y}'A\underline{Y} : AX = 0\}$, from Seely (1971), a necessary and sufficient condition for $\sum_{i=1}^{t} \nu_i c_i$ to be $\overline{N}$-estimable is the existence of a vector $\underline{\alpha}$ which satisfies the following equations:

$$
\begin{aligned}
&\sum_{i=1}^{t} \operatorname{tr}(NV_1NV_iN)\alpha_i = \nu_1 \\
&\sum_{i=1}^{t} \operatorname{tr}(NV_2NV_iN)\alpha_i = \nu_2 \\
&\qquad\vdots \\
&\sum_{i=1}^{t} \operatorname{tr}(NV_tNV_iN)\alpha_i = \nu_t
\end{aligned}
\tag{6.3.1}
$$

In this case, $P = \frac{1}{r} D'D = \frac{1}{r} T = E_3 + E_4$ and $N = I - P = I - \frac{1}{r} T =$ $= -\frac{1}{\phi}(E_0 + E_1) + \frac{(\phi + 1)}{\phi} E_2 + \frac{1}{\phi} E_3 + E_5 + E_6$. We will use the base matrices as defined in (2.1). If $\lambda = 1$ (and hence, $S = B + T - I$), the equations (6.3.1) thus reduce to

$$
\begin{aligned}
&\frac{\phi^2}{(\phi + 1)^2}(v - 1)\ \alpha_2 = \nu_2 \\
&(n - b - v + 1)\ \alpha_5 = \nu_5 \\
&(b - v)\ \alpha_6 = \nu_6.
\end{aligned}
\tag{6.3.2}
$$

(See Appendix F.) The only $\bar{N}$-estimable function is of the form $\nu_2 c_2 + \nu_5 c_5 + \nu_6 c_6$ where ν_2, ν_5 and ν_6 are real numbers. An unbiased estimator for this function is

$$\sum_{i=1}^{6} \alpha_i \underline{Y}'NV_iN\underline{Y} = \frac{(\phi + 1)^2 \nu_2}{\phi^2(\nu - 1)} \underline{Y}' \left[-\left(\frac{1}{\phi + 1}\right)(E_0 + E_1) + E_2 + \left(\frac{1}{\phi + 1}\right) E_3 \right] \underline{Y} +$$

$$+ \frac{\nu_5}{n-b-v+1} Y'E_5Y + \frac{\nu_6}{b-v} \underline{Y}'E_6\underline{Y}.$$

Of course, for the fixed model, the b.l.u.e. of an estimable linear function of the treatment (block) parameters is equivalent to the simple least squares estimator of the function. For this model, however, the b.l.u.e. does not necessarily equal the l.s.e. If $\hat{M}$ is the b.l.u.e. of $E(\underline{Y})$ and M* is the l.s.e. of $E(\underline{Y})$, the difference between $\hat{M}$ and M* is

$$M^* - \hat{M} = PV(I - P)\,[(I - P)V\,(I - P)]^+\,(I - P)\underline{Y}$$

where V is the covariance matrix (possibly singular) and H^+ denotes the Moore-Penrose inverse of H. (See Baksalary and Kala (1980).) We only determine this difference for the case where $V = \sum_{i=1}^{t} c_i V_i \in \mathcal{R}_7$; minor modifications are necessary for covariance matrices in $\mathcal{R}_6$, $\mathcal{P}_7$ or $\mathcal{P}_8$. For this case

$$(I - P)V\,(I - P) = c_2\left[-\left(\frac{1}{\phi + 1}\right)(E_0 + E_1) + E_2 + \left(\frac{1}{\phi + 1}\right) E_3\right] + c_5E_5 + c_6E_6.$$

If $c_i^+ = \frac{1}{c_i}$ for $c_i \neq 0$ (=0 otherwise), then $[(I - P)V(I - P)]^+$

$$= c_2^+\left[-\frac{(\phi + 1)}{\phi^2}(E_0 + E_1) + \frac{(\phi + 1)^2}{\phi^2} E_2 + \frac{\phi + 1}{\phi^2} E_3\right] + c_5^+E_5 + c_6^+E_6.$$

Therefore,

$$M^* - \hat{M} = PV(I - P)[(I - P)V(I - P)]^+ (I - P)\,\underline{Y}$$

$$= (E_3 + E_4)\, V \left[-\frac{1}{\phi}(E_0 + E_1) + \left(\frac{\phi + 1}{\phi}\right) E_2 + \frac{1}{\phi} E_3 + E_5 + E_6\right] \times$$

$$[(I - P)V(I - P)]^+ (I - P)\underline{Y}$$

$$= \left(c_1 + \frac{c_2}{\phi + 1}\right) (E_1 - E_3)\, [(I - P)V(I - P)]^+ (I - P)\underline{Y}$$

$$= \left(c_1 + \frac{c_2}{\phi + 1}\right) c_2^+ \left(\frac{\phi + 1}{\phi}\right) (E_1 - E_3)\underline{Y}$$

$$= \left(c_1 + \frac{c_2}{\phi + 1}\right) c_2^+ \left(\frac{\phi + 1}{\phi}\right) \left(\frac{TB}{r - \lambda} - \frac{T}{r} - \frac{\phi}{rv} G\right)\underline{Y}$$

where $c_2^+ = \frac{1}{c_2}$ if $c_2 \neq 0$ (= 0 otherwise). If $c_2 = 0$, then $M^* = \hat{M}$.

VII. CONCLUSION

The major part of this dissertation is concerned with determining the commutant algebra for the matrix representation reflecting the permutation of the incident point-hyperplane pairs under the action of the full collineation group on a desarguesian finite projective geometry or a desarguesian finite affine geometry. These geometries possess a great deal of symmetry; other BIB designs most likely do not. Thus, if the full automorphism group for a design were known, the dimension of the commutant algebra is likely to be greater than that of the appropriate relationship algebra or extended relationship algebra. Many designs have automorphism groups whose order is less than the number of experimental units n, in which case the group is intransitive on the flags. The dimension of the commutant algebra is often significantly increased.

We have been working with *Steiner Triple Systems*, i.e., block designs with $k = 3$, $\lambda = 1$. If $v = 13$, there are two non-isomorphic systems, each having an automorphism group which is intransitive on the flags. If $v = 15$, there are eighty non-isomorphic systems, most having flag-intransitive automorphism groups. The dimension of the commutant algebra is often so large, the problem is of little interest. We do intend to further study the commutant algebras associated with different designs. For example, we wish to examine the effects of resolvability and of repeated blocks and the case where we have a partially balanced

incomplete block design. We also wish to see if our work with the desarguesian geometries will help us for different types of geometries.

Concerning estimation, we have seen that for the fixed model, the b.l.u.e. of an estimable linear function of the treatment (block) parameters equals the l.s.e. of that function, if the covariance matrix is in R_6, R_7, P_7 or P_8. As the algebra increases in size, this may very well not be true; certainly it is not true for the random blocks model. Much estimation and analysis work remains to be done.

BIBLIOGRAPHY

Baksalary, J. K., and Kala, R. (1980), "A new bound for the Euclidean norm of the difference between the least squares and the best linear unbiased estimators," The Annals of Statistics, *8*, 679-681.

Carmichael, R. D. (1937), Introduction to the Theory of Groups of Finite Order, Ginn and Company.

Cochran, W. G., and Cox, G. M. (1950), Experimental Designs, New York, John Wiley and Sons, Inc.

Dembowski, P. (1968), Finite Geometries, New York, Springer-Verlag, Inc.

Drygas, H. (1977), "Best quadratic unbiased estimation in variance-covariance component models," Mathematische Operationsforschung und Statistik Series Statistics, *8*, 211-230.

Dubenko, T. I., Sysoev, L. P., Shaikin, M. E. (1976), "Sufficient Statistics and estimates of two specially structured covariance matrices for two experimental design models," Avtomatika i Telemekhanika (Russian), *4*, 25-34, translated in Automation and Remote Control (1976), 492-500.

Dubenko, T. I., Sysoev, L. P., Shaikin, M. E. (1976), "Symmetry properties and characterization of covariance matrices in the experiment design problem with randomized blocks," Avtomatika i Telemekhanika (Russian), *3*, 73-82, translated in Automation and Remote Control (1976), 378-386.

Gleeson, A. C., and McGilchrist, C. A. (1978), "An Introduction to Relationship Algebra," Communications in Statistics, A, Theory and Methods, *7*(11), 1053-1078.

Gnot, S., Klonecki, W., and Zmyślony, R. (1977), "Uniformly minimum variance unbiased estimation in various classes of estimators, I," Mathematische Operationsforschung und Statistik Series Statistics, *8*, 199-210.

Hall, Jr., M. (1967), Combinatorial Theory, Waltham, Mass., Blaisdell Publishing Co.

Hall, Jr., M. (1959), The Theory of Groups, New York, The Macmillan Co.

Halmos, P. R. (1958), Finite Dimensional Vector Spaces, second edition, Princeton, D. Van Nostrand Company.

Hannan, E. J. (1965), Group Representations and Applied Probability, Methuen's Mimeographs on Applied Probability and Statistics, Supplementary Review Series in Applied Probability, J. Gani, ed., Vol. 3, London, Methuen and Co., Ltd.

James, A. T. (1957), "The relationship algebra of an experimental design," Annals of Mathematical Statistics, *28*, 993-1002.

Kempthorne, Oscar (1952), The Design and Analysis of Experiments, New York, John Wiley and Sons, Inc.

Kleffé, J. (1977), "Invariant methods for estimating variance components in mixed linear models," Mathematische Operationsforschung und Statistik Series Statistics, *8*, 233-250.

Kruskal, W. (1968), "When are Gauss-Markov and least squares estimators identical? A coordinate-free approach," The Annals of Mathematical Statistics, *39*, 70-75.

Ledermann, W. (1977), Introduction to Group Characters, Cambridge, Cambridge University Press.

Littlewood, D. E. (1950), The Theory of Group Characters and Matrix Representations of Groups, second edition, Oxford, Clarendon Press.

Liu, C. L. (1968), Introduction to Combinatorial Mathematics, New York, McGraw-Hill Book Company.

Mann, H. B. (1960), "The algebra of a linear hypothesis," The Annals of Mathematical Statistics, *31*, 1-15.

McLaren, A. D. (1963), "On group representations and invariant stochastic processes," Proceedings of the Cambridge Philosophical Society, *59*, 431-450.

Mitra, S. K., and Moore, B. J. (1973), "Gauss-Markov estimation with an incorrect dispersion matrix," Sankhya, Series A, *35*, 139-152.

Mitra, S. K., and Thorne, B. J. M. (1976), "Linear Estimation and General Linear Models," Essays in Probability and Statistics, S. Ikeda and others, ed., Shinko Tsusko Co., Tokyo, 189-201.

Pukelsheim, F. (1976), "Estimating variance components in linear models," Journal of Multivariate Analysis, *6*, 626-629.

Pukelsheim, F., and Styan, G. P. H. (1978), "Nonnegative definiteness of the estimated dispersion matrix in a multivariate linear model," Technical Report No. 32, Office of Naval Research.

Rao, C. R. (1973), Linear Statistical Inference and its Applications, second edition, New York, John Wiley and Sons, Inc.

Scheffe, H. (1959), The Analysis of Variance, New York, John Wiley and Sons, Inc.

Schur, I. (1933), "Zur Theorie der einfach transitiven Permutationsgruppen," Akademie der Wissenschaften, Berlin, Physikalisch-Mathematisch Klasse, Sitzungsberichte.

Seely, J. (1969), "Estimation in finite dimensional vector spaces with application to the mixed linear model," Ph.D. thesis, Iowa State University, Ames.

Seely, J. (1971), "Quadratic subspaces and completeness," The Annals of Mathematical Statistics, *42*, 710-721.

Serre, J. P. (1971), Representations Linéaires des Groupes Finis, Paris, Hermann (translated by Leonard L. Scott as Linear Representations of Finite Groups (1977), Graduate Texts in Mathematics #42, New York, Springer-Verlag).

Sysoev, L. P., and Shaikin, M. E. (1976), "Algebraic methods of investigating the correlation connections in incompletely balanced block schemes for experiment designs. I. Characterization of covariance matrices in the case of asymmetric block schemes," Avtomatika i Telemekhanika (Russian), *5*, 64-73 (translated as Automation and Remote Control, *37*, 696-704). "II. Relationship algebras and characterization of covariance matrices in the case of symmetric block schemes," Avtomatika i Telemekhanika (Russian), *7*, 57-67 (translated as Automation and Remote Control, *37*, 1022-1031). "III. Sufficient statistics and estimates of covariance matrices," Avtomatika i Telemekhanika (Russian), *9*, 62-68 (translated as Automation and Remote Control, *37*, 1349-1354).

Sysoev, L. P. (1977), "Properties of certain classes of experimental block designs and correlation characteristics of observations," Avtomatika i Telemekhanika (Russian), *8*, 41-53 (translated as Automation and Remote Control, *38*, 1145-1156).

Van der Waerden, B. L. (1970), Algebra, Vol. 2, New York, Frederick Ungar Publishing Co. (translated by John R. Schulenberger from Algebra, fifth edition, Berlin, Springer-Verlag).

Watson, G. S. (1967), "Linear Least Squares Regression," The Annals of Mathematical Statistics, *38*, 1679-1699.

Watson, G. S. (1972), "Prediction and the efficiency of least squares," Biometrika, *59*, 91-98.

Weyl, H. (1952), Symmetry, Princeton, Princeton University Press.

Wielandt, H. (1964), Finite Permutation Groups, New York, Academic Press, Inc. (translated from the German by R. Bercov).

Yates, F. (1940), "The recovery of inter-block information in balanced incomplete block designs," Annals of Eugenics, *10*, 317-325.

Zyskind, G. (1975), "Error Structures, Projections and Conditional Inverses in Linear Model Theory," A Survey of Statistical Design and Linear Models, J. N. Srivastava, editor, North-Holland Publishing Company, 647-663.

Zyskind, G. (1967), "On canonical forms, non-negative covariance matrices and best and simple least squares linear estimators in linear models," Annals of Mathematical Statistics, *38*, 1092-1109.

APPENDIX A

Traditional Analysis of a Balanced Incomplete Block Design

Intra-block analysis of a BIB design refers to the standard least squares analysis based on the normal theory model $E(\underline{Y}) = \mu \underline{j}_n + D'\underline{\tau} + L'\underline{\beta}$, $V(\underline{Y}) = \sigma^2 I_n$, $\sum_i \tau_i = 0$, $\sum_j \beta_j = 0$. The general ANOVA table for the design is as follows:

SOURCE	SS	DF	E(MS)
Blocks, ignoring treatments			
Treatment component	$\frac{1}{k(r-\lambda)} \sum_i T_i^2 -$ $-(\phi+1)C$	v-1	
Remainder	$\frac{1}{k} \sum_j h_j^2 -$ $- \frac{1}{k(r-\lambda)} \sum_i T_i^2 +$ $+\phi C$	b-v	
Total	$\frac{1}{k} \sum_j h_j^2 - C$	b-1	
Treatments, eliminating blocks	$\frac{(\phi+1)}{\phi r} \sum_i G_i^2$	v-1	$\sigma^2 + \frac{r\phi}{(\phi+1)} \sigma_T^2$
Error	by subtraction	n-v-b+1	σ^2
Total	$\sum_{i,j} Y_{ij}^2 - C$	n-1	

$C = n^{-1} \left(\sum_{i,j} Y_{ij} \right)^2$ is the correction for the grand mean, $G_i = g_i - k^{-1} T_i$ is the adjusted treatment total, $\frac{\phi}{\phi+1} = \frac{\lambda v}{rk}$ is the *efficiency factor* of the design, $\sigma_T^2 = (v-1)^{-1} \sum_i \tau_i^2$ and g_i, h_j and T_i are defined in Chapter II. (The ANOVA table will be amended for a symmetric BIB

design and for a resolvable BIB design. See Kempthorne (1952).) Treatment effects are estimated by $\hat{\tau}_i = \frac{(\phi+1)}{\phi r} G_i$; treatment contrasts $\psi = \sum_i c_i \tau_i$, $\sum_i c_i = 0$, are estimated by $\hat{\psi} = \sum_i c_i \hat{\tau}_i$ where $\text{var}(\hat{\psi}) = \sigma^2 \frac{(\phi+1)}{\phi r} \sum_i c_i^2$. Under the fixed effects model, these treatment contrast estimates are optimal.

Recovery of *intrablock information* is helpful if we assume the random block effect model under normality. $\underline{Y}$ is normally distributed with $E(\underline{Y}) = \mu \underline{j}_n + D'\underline{\tau}$, $\text{var}(\underline{Y}) = \sigma^2 I_n + \sigma_1^2 L'L$ and $\sum_i \tau_i = 0$. (We use a technique invented by Yates (1940); this assumes $b > v$.) Now, in addition to the intrablock estimates $\{\hat{\tau}_i\}$ which have the same distribution under the random block model as the fixed model, we may determine interblock estimates $\{\hat{\tau}_i'\}$, unbiased estimates for $\underline{\tau}$ which are statistically independent from $\{\hat{\tau}_i\}$. These estimates are derived by treating the block totals as observations in a fixed effects model; we find $\hat{\tau}_i' = (r-\lambda)^{-1} (T_i - \frac{r}{b} \sum_j h_j)$. Any contrast estimator $\hat{\psi}' = \sum_i c_i \hat{\tau}_i'$ has variance $\text{var}(\hat{\psi}') = \frac{(\phi+1)}{r} (\sigma^2 + k\sigma_1^2)^{-1} \sum_i c_i^2$. If we define $w = \frac{\phi r}{(\phi+1)\sigma^2}$ and $w' = \frac{(r-\lambda)}{k(\sigma^2+k\sigma_1^2)} = \frac{r}{(\phi+1)(\sigma^2+k\sigma_1^2)}$, $\frac{\hat{w\psi} + \hat{w'\psi}'}{w + w'}$ is an unbiased estimator of ψ with minimum variance $\frac{1}{(w+w')} \sum c_i^2$. We use $\frac{\phi r}{(\phi+1)\text{MSE}}$ to estimate w and $r(n-v)(\phi+1)^{-1}[k(b-1)\text{MS}_{\text{bl.el.tr.}} - (v-k)\text{MSE}]^{-1}$ to estimate w'. Inferences about $\{\tau_i\}$ are made by assuming $\hat{w}$ and $\hat{w}'$ are equal to w and w', respectively.

APPENDIX B

The subgroup of $PC(2, p^r)$ which fixes the point (1 0 0) and the line (0 0 1) consists of the point transformations of the form (for a given $m = p^u$, $u = 0, 1, \ldots, r-1$):

$$A\underset{\sim}{x}^m = \begin{pmatrix} 1 & a_{01} & a_{02} \\ 0 & a_{11} & a_{12} \\ 0 & 0 & a_{22} \end{pmatrix} \begin{pmatrix} x_0^m \\ x_1^m \\ x_2^m \end{pmatrix} = \alpha \begin{pmatrix} x_0^* \\ x_1^* \\ x_2^* \end{pmatrix},$$

$$\alpha(\neq 0) \in GF(s),\ a_{ij} \in GF(s)$$

where a_{11} and a_{22} are non-zero. The flag type pairs found in Table 7 and the conditions under which they are fixed are:

a) Point (1 0 0) Line (0 0 1)

Fixed for all matrices A in the stabilizer

There are $(s-1)^2 s^3$ collineations fixing this flag.

b) Point (1 0 0) Line (0 1 a), $a \in GF(s)$

$a_{12} + a_{22}a = a_{11}a^m$

There are $(s-1)^2 s^2$ collineations fixing this flag.

c) Point (c 1 0); $c \in GF(s)$ Line (0 0 1)

$c^m + a_{01} = a_{11}c$

There are $(s-1)^2 s^2$ collineations fixing this flag.

d) Point (c 1 0); $c \in GF(s)$ Line (1 -c b), $b \in GF(s)$

$c^m + a_{01} = a_{11}c$

$a_{02} - a_{12}c + a_{22}b = b^m$

There are $(s-1)^2 s$ collineations fixing this flag.

e) Point (c d 1); c, d $\in$ GF(s) Line (0 1 -d)

$c^m + a_{01}d^m + a_{02} = a_{22}c$

$a_{11}d^m + a_{12} = a_{22}d$

There are $(s - 1)^2 s$ collineations fixing this flag.

f) Point (c d 1); c, d, $\in$ GF(s) Line (1 a -c-ad); a $\in$ GF(s)

$c^m + a_{01}d^m + a_{02} = a_{22}c$ $a_{11}d^m + a_{12} = a_{22}d$

$a_{01} + a_{11}a = a^m$

There are $(s - 1)^2$ collineations fixing this flag.

APPENDIX C

The subgroup of $PC(k, p^r)$, $k \geq 3$ which fixes the point $(1, 0, 0, \ldots, 0)$ and the hyperplane $(0, 0, \ldots, 0, 1)$ consists of the transformations of the form (for a given $m = p^u$, $u = 0, 1, \ldots, r-1$):

$$A\underline{x}_m = \begin{pmatrix} 1 & \underline{a}_1' & a_{0k} \\ \underline{0} & A_1 & \underline{a}_2 \\ 0 & \underline{0}' & a_{kk} \end{pmatrix} \begin{pmatrix} x_0^m \\ x_1^m \\ \cdot \\ \cdot \\ \cdot \\ x_k^m \end{pmatrix} = \alpha \begin{pmatrix} x_0^* \\ x_1^* \\ \cdot \\ \cdot \\ \cdot \\ x_k^* \end{pmatrix},$$

$$\alpha(\neq 0) \in GF(s),\ a_{ij} \in GF(s)$$

where $\underline{a}_1' = (a_{01}\ \ a_{02}\ \cdots\ a_{0,\,k-1})$, $\underline{a}_2 = (a_{1k}\ \ a_{2k}\ \cdots\ a_{k-1,\,k})'$,

$$A_1 = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1,\,k-1} \\ a_{21} & a_{22} & \cdots & a_{2,\,k-1} \\ & & \cdot & \\ & & \cdot & \\ & & \cdot & \\ a_{k-1,\,1} & a_{k-1,\,2} & \cdots & a_{k-1,\,k-1} \end{pmatrix}$$

and A is non-singular. The flag type pairs found in Table 8 and the conditions under which they are fixed are:

a) Point $(1, 0, 0, \ldots, 0)$ Hyperplane $(0, 0, \ldots, 0, 1)$

Fixed for all A in the stabilizer

There are $(s-1)s^{2k-1}\Phi(s, k-1)$ collineations fixing this flag, where $\Phi(s, k-1) = \prod_{i=0}^{k-2} (s^{k-1} - s^i)$.

b) Point $(1, 0, 0, \ldots, 0)$ Hyperplane $(0, b_1, b_2, \ldots, b_k)$; $b_i \in GF(s)$; $b_1, b_2, \ldots, b_{k-1}$ not all zero

$$A_1' \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{k-1} \end{pmatrix} = \gamma \begin{pmatrix} b_1^m \\ b_2^m \\ \vdots \\ b_{k-1}^m \end{pmatrix}, \quad \gamma(\neq 0) \in GF(s)$$

$$a_{1k}b_1 + a_{2k}b_2 + \ldots + a_{kk}b_k = \gamma b_k^m$$

The number of matrices A_1 satisfying the first condition is the same for any choice of $b_1, b_2, \ldots, b_{k-1}$ (not all zero). This is true since the condition above is equivalent to the condition

$$(A_1')^{-1} \begin{pmatrix} b_1^m \\ b_2^m \\ \vdots \\ b_{k-1}^m \end{pmatrix} = \beta \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{k-1} \end{pmatrix}, \quad \beta(\neq 0) \in GF(s).$$

In a sense, if we think of $(b_1, b_2, \ldots, b_{k-1})$ as a hyperplane vector in a $(k-2)$-dimensional space, we are just counting the number of elements in the subgroup which fixes the hyperplane, which will be the same for any choice of $b_1, b_2, \ldots, b_{k-1}$. To count our matrices, we will therefore make a choice of $b_1 = 1, b_2 = b_3 = \ldots = b_{k-1} = 0$. Our conditions thus reduce

to

$$a_{11} = \nu;\ a_{12} = a_{13} = \cdots = a_{1,\,k-1} = 0;\ a_{1k}b_1 + a_{kk}b_k = \gamma b_k^{\,m}.$$

There are $(s-1)^2 s^{3k-4}\Phi(s, k-2)$ collineations fixing this flag. This is true since there are $s-1$ possibilities for both a_{11} and a_{kk}, s possibilities for each of $a_{01}, a_{02}, \ldots, a_{0k}, a_{2k}, a_{3k}, \ldots, a_{k-1,\,k}, a_{21}, a_{31}, \ldots, a_{k-1,\,1}$ and $\Phi(s, k-2) = \prod_{i=0}^{k-3} (s^{k-2} - s^i)$ possibilities for

$$\begin{pmatrix} a_{22} & a_{23} & \cdots & a_{2,\,k-1} \\ a_{32} & a_{33} & \cdots & a_{3,\,k-1} \\ & & \vdots & \\ a_{k-1,\,2} & a_{k-1,\,3} & \cdots & a_{k-1,\,k-1} \end{pmatrix}.$$

c) Point $(c_1, c_2, \ldots, c_k, 0)$; Hyperplane $(0, 0, \ldots, 0, 1)$

$c_i \in GF(s)$;

$c_2, c_3, \ldots, c_k$ not all zero

$$A_1 \begin{pmatrix} c_2^{\,m} \\ c_3^{\,m} \\ \vdots \\ c_k^{\,m} \end{pmatrix} = \alpha \begin{pmatrix} c_2 \\ c_3 \\ \vdots \\ c_k \end{pmatrix}, \quad \alpha(\neq 0) \in GF(s)$$

$$c_1^{\,m} + a_{01}c_2^{\,m} + a_{02}c_3^{\,m} + \cdots + a_{0,\,k-1}c_k^{\,m} = \alpha c_1$$

The count will be the same for any choice of $c_2, c_3, \ldots, c_k$ (not all zero); therefore we choose $c_2 = 1$, $c_3 = c_4 = \cdots = c_k = 0$.

Our conditions thus reduce to

$$a_{11} = \alpha;\ a_{21} = a_{31} = \cdots = a_{k-1,\,1} = 0;\ c_1^{\,m} + a_{01}c_2^{\,m} = \alpha c_1;$$

$$\alpha(\neq 0) \in GF(s).$$

There are $(s-1)^2 s^{3k-4}\Phi(s, k-2)$ collineations which fix the flag.

d) Point $(c_1, c_2, \ldots, c_k, 0)$; Hyperplane $(0, b_1, b_2, \ldots, b_k)$;

$c_i \in GF(s)$; $b_i \in GF(s)$;

$c_2, c_3, \ldots, c_k$ not all zero $b_1, b_2, \ldots, b_{k-1}$ not all zero;

$$c_2b_1 + c_3b_2 + \cdots + c_kb_{k-1} = 0$$

$$A_1 \begin{pmatrix} c_2^{\,m} \\ c_3^{\,m} \\ \vdots \\ c_k^{\,m} \end{pmatrix} = \alpha \begin{pmatrix} c_2 \\ c_3 \\ \vdots \\ c_k \end{pmatrix};\quad A_1' \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{k-1} \end{pmatrix} = \gamma \begin{pmatrix} b_1^{\,m} \\ b_2^{\,m} \\ \vdots \\ b_{k-1}^{\,m} \end{pmatrix};\quad \alpha, \gamma(\neq 0) \in GF(s)$$

$$c_1^{\,m} + a_{01}c_2^{\,m} + a_{02}c_3^{\,m} + \cdots + a_{0,\,k-1}c_k^{\,m} = \alpha c_1$$

$$a_{1k}b_1 + a_{2k}b_2 + \cdots + a_{kk}b_k = \gamma b_k^{\,m}$$

1) For $k \geq 4$, the two matrix equations are essentially equivalent to equations for the stabilizer of the flag consisting of the point $(c_2, c_3, \ldots, c_k)$ and the hyperplane $(b_1, b_2, \ldots, b_{k-1})$ in a $(k-2)$-dimensional space. Thus the number of matrices A_1 which satisfy the two matrix equations will be constant for any choice of $(c_2, c_3, \ldots, c_k)$ and $(b_1, b_2, \ldots, b_{k-1})$ (consistent with our restrictions on the values). We therefore choose $c_2 = b_{k-1} = 1$, $c_3 = c_4 = \cdots = c_k = b_1 = b_2 = \cdots = b_{k-2} = 0$.

Our conditions therefore reduce to

$a_{11} = \alpha$; $a_{k-1,\,k-1} = \gamma$; $a_{21} = a_{31} = \cdots = a_{k-1,\,1} = a_{k-1,\,2} =$

$= a_{k-1,\,3} = \cdots = a_{k-1,\,k-2} = 0$;

$c_1^m + a_{01}c_2^m = \alpha c_1$; $a_{k-1,\,k}b_{k-1} + a_{kk}b_k = \gamma b_k^m$; $\alpha,\ \gamma(\neq 0) \in GF(s)$.

There are $(s-1)^3 s^{4k-8}\Phi(s, k-3)$ collineations which fix the flag.

2) For k = 3, the conditions are:

$$A_1 \begin{pmatrix} c_2^m \\ c_3^m \end{pmatrix} = \alpha \begin{pmatrix} c_2 \\ c_3 \end{pmatrix};\quad A_1' \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \gamma \begin{pmatrix} b_1^m \\ b_2^m \end{pmatrix};\quad \alpha,\ \gamma(\neq 0) \in GF(s);$$

$$c_1^m + a_{01}c_2^m + a_{02}c_3^m = \alpha c_1;\ a_{13}b_1 + a_{23}b_2 + a_{33}b_3 = \gamma b_3^m$$

There are $(s-1)^3 s^4$ collineations which fix the flag.

e) Point $(c_1, c_2, \ldots, c_k, 0)$; Hyperplane $(1, b_1, b_2, \ldots, b_k)$;

$c_i \in GF(s)$; $b_i \in GF(s)$;

$c_2, c_3, \ldots, c_k$ not all zero $c_1 + c_2b_1 + \cdots + c_kb_{k-1} = 0$

$$\begin{pmatrix} a_{01} \\ a_{02} \\ \cdot \\ \cdot \\ \cdot \\ a_{0,\,k-1} \end{pmatrix} + A_1' \begin{pmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_{k-1} \end{pmatrix} = \begin{pmatrix} b_1^m \\ b_2^m \\ \cdot \\ \cdot \\ \cdot \\ b_{k-1}^m \end{pmatrix},\quad A_1 \begin{pmatrix} c_2^m \\ c_3^m \\ \cdot \\ \cdot \\ \cdot \\ c_k^m \end{pmatrix} = \alpha \begin{pmatrix} c_2 \\ c_3 \\ \cdot \\ \cdot \\ \cdot \\ c_k \end{pmatrix}$$

$$a_{0k} + a_{1k}b_1 + a_{2k}b_2 + \cdots + a_{kk}b_k,\ \alpha(\neq 0) \in GF(s)$$

$(c_1^m + a_{01}c_2^m + \cdots + a_{0,\,k-1}c_k^m = \alpha c_1$ is a linear combination of the other conditions. Here we use the fact that

$c_1 + c_2b_1 + \ldots + c_kb_{k-1} = 0.$) There are $(s - 1)s^{k-2}\Phi(s, k - 2)$ possibilities for A_1. When A_1 is determined, $a_{01}, a_{02}, \ldots, a_{0, k-1}$ are completely determined. Therefore, there are $(s - 1)^2s^{2k-3}\Phi(s, k-2)$ collineations fixing the flag.

f) Point $(c_1, c_2, \ldots, c_k, 1)$, $c_i \in GF(s)$

a) $(0, b_1, b_2, \ldots, b_k)$; $b_i \in GF(s)$; $b_1, b_2, \ldots, b_{k-1}$ not all zero; $c_2b_1 + c_3b_2 + \ldots + c_kb_{k-1} + b_k = 0$

$$A_1 \begin{pmatrix} c_2^m \\ c_3^m \\ \cdot \\ \cdot \\ \cdot \\ c_k^m \end{pmatrix} + \begin{pmatrix} a_{1k} \\ a_{2k} \\ \cdot \\ \cdot \\ \cdot \\ a_{k-1, k} \end{pmatrix} = a_{kk} \begin{pmatrix} c_2 \\ c_3 \\ \cdot \\ \cdot \\ \cdot \\ c_k \end{pmatrix}, \quad A_1' \begin{pmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_{k-1} \end{pmatrix} = \gamma \begin{pmatrix} b_1^m \\ b_2^m \\ \cdot \\ \cdot \\ \cdot \\ b_{k-1}^m \end{pmatrix}$$

$$c_1^m + a_{01}c_2^m + a_{02}c_3^m + \ldots + a_{0, k-1}c_k^m + a_{0k} = a_{kk}c_1$$

$\gamma (\neq 0) \in GF(s)$

($a_{1k}b_1 + a_{2k}b_2 + \ldots + a_{kk}b_k = \gamma b_k^m$ is a linear combination of the other conditions.) There are $(s - 1)s^{k-2}\Phi(s, k - 2)$ possibilities for A_1'. When A_1' and a_{kk} are determined, $a_{1k}, a_{2k}, \ldots, a_{k-1, k}$ are completely determined. There are $(s - 1)^2s^{2k-3}\Phi(s, k - 2)$ collineations fixing the flag.

g) Point $(c_1, c_2, \ldots c_k, 1)$, $c_i \in GF(s)$ Hyperplane $(1, b_1, b_2, \ldots, b_k)$; $b_i \in GF(s)$;

$$c_1 + c_2b_1 + c_3b_2 + \ldots + c_kb_{k-1} + b_k = 0$$

$$\begin{pmatrix} a_{01} \\ a_{02} \\ \vdots \\ a_{0,\,k-1} \end{pmatrix} + A_1 \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{k-1} \end{pmatrix} = \begin{pmatrix} b_1^m \\ b_2^m \\ \vdots \\ b_{k-1}^m \end{pmatrix}, \; A_1 \begin{pmatrix} c_2^m \\ c_3^m \\ \vdots \\ c_k^m \end{pmatrix} + \begin{pmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a_{k-1,\,k} \end{pmatrix} = a_{kk} \begin{pmatrix} c_2 \\ c_3 \\ \vdots \\ c_k \end{pmatrix}$$

$$a_{0k} + a_{1k}b_1 + a_{2k}b_2 + \ldots + a_{kk}b_k = b_k^m$$

There are $\Phi(s, k-1)$ possibilities for A_1. Once A_1 and a_{kk} are determined, all of the remaining values of a_{ij} are completely determined. Therefore there are $(s-1)\Phi(s, k-1)$ collineations fixing the flag.

APPENDIX D

The subgroup of $EC(2, p^r)$ which fixes the point (0, 0) and the line (0 0 1) ($x_2 = 0$) consists of the transformations (for a given $m = p^u$, $u = 0, 1, \ldots, r - 1$):

$$A\underset{\sim}{x}_m = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} \begin{pmatrix} x_1^m \\ x_2^m \end{pmatrix} = \begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix}$$

such that a_{11}, $a_{22}(\neq 0) \in GF(s)$. The flag type pairs found in Table 9 and the conditions under which they are fixed are:

a) Point (0, 0) Line (0 0 1)

Fixed for all matrices A in the stabilizer

There are $(s - 1)^2 s$ collineations which fix the flag.

b) Point (0, 0) Line (0 1 a), $a \in GF(s)$

$a_{12} + a_{22}a = a_{11}a^m$

There are $(s - 1)^2$ collineations which fix the flag.

c) Point (c, 0), Line (0 0 1)

$c(\neq 0) \in GF(s)$

$a_{11}c^m = c$

There are $(s - 1)s$ collineations which fix the flag.

d) Point (c, 0), Line $(1 \ -c^{-1} \ b)$, $b \in GF(s)$

$c(\neq 0) \in GF(s)$

$a_{11}c^m = c$

$a_{22}b = b^m + a_{12}c^{-1}$

There are $s - 1$ collineations fixing the flag.

e) Point (c, d); Line (0 1 a), $a = -cd^{-1}$

$c, d \in GF(s)$;

$d \neq 0$

$a_{11}c^m + a_{12}d^m = c$

$a_{22}d^m = d$

There are s - 1 collineations which fix the flag.

f) Point (c, d); Line (1 0 a), $a = -d^{-1}$

$c, d \in GF(s)$;

$d \neq 0$

$a_{11}c^m + a_{12}d^m = c$

$a_{22}d^m = d$

There are s - 1 collineations which fix the flag.

g) Point (c, d); Line (1 a b), $a(\neq 0) \in GF(s)$,

$c, d \in GF(s)$; $b = -(1 + ac)d^{-1}$

$d \neq 0$

$a_{11}c^m + a_{12}d^m = c$

$a_{22}d^m = d$

$a_{11}a = a^m$

There is 1 collineation which fixes the flag.

APPENDIX E

The subgroup of $EC(k, p^r)$, $k \geq 3$ which fixes the point $(0, 0, \ldots, 0)$ and the hyperplane $(0, 0, \ldots, 0, 1)$ consists of the transformations of the form (for a given $m = p^u$, $u = 0, 1, \ldots, r - 1$):

$$A\underline{x}_m = \begin{pmatrix} A_1 & \underline{a} \\ \underline{0}' & a_{kk} \end{pmatrix} \begin{pmatrix} x_1^m \\ x_2^m \\ \cdot \\ \cdot \\ \cdot \\ x_k^m \end{pmatrix} = \begin{pmatrix} x_1^* \\ x_2^* \\ \cdot \\ \cdot \\ \cdot \\ x_k^* \end{pmatrix}, \quad a_{ij} \in GF(s),$$

where

$$A_1 = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1,\,k-1} \\ a_{21} & a_{22} & \cdots & a_{2,\,k-1} \\ & & \vdots & \\ a_{k-1,\,1} & a_{k-1,\,2} & \cdots & a_{k-1,\,k-1} \end{pmatrix}, \quad |A_1| \neq 0,$$

$$\underline{a}' = (a_{1k} \quad a_{2k} \quad \cdots \quad a_{k-1,\,k}), \quad a_{kk} \neq 0.$$

The flag type pairs found in Table 10 and the conditions under which they are fixed are:

a) Point $(0, 0, \ldots, 0)$ Hyperplane $(0, 0, \ldots, 0, 1)$

Fixed for all A in the stabilizer

There are $(s - 1)s^{k-1}\Phi(s, k - 1) = (s - 1)s^{k-1} \prod_{i=0}^{k-2} (s^{k-1} - s^i)$ collineations which fix the flag ($s - 1$ possibilities for a_{kk}, s^{k-1} possibilities for $\underline{a}$ and $\Phi(s, k - 1)$ possibilities for A_1).

b) Point $(0, 0, \ldots, 0)$ Hyperplane $(0, b_1, b_2, \ldots, b_k)$;

$b_i \in GF(s)$;

$b_1, b_2, \ldots b_{k-1}$ not all zero

$$A_1' \begin{pmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_{k-1} \end{pmatrix} = \gamma \begin{pmatrix} b_1^m \\ b_2^m \\ \cdot \\ \cdot \\ \cdot \\ b_{k-1}^m \end{pmatrix}, \quad \gamma (\neq 0) \in GF(s)$$

$$a_{1k}b_1 + a_{2k}b_2 + \ldots + a_{kk}b_k = \gamma b_k^m$$

The number of matrices A_1 which satisfy the matrix equation is constant for any choice of $b_1, b_2, \ldots, b_{k-1}$ not all zero. This is true since the problem of determining the number of matrices A_1 is equivalent up to a factor of $s - 1$ to the problem of determining the number of elements in the stabilizer (in the group $PC(k - 2, s)$) of a hyperplane (or a point) in the finite projective space $PG(k - 2, s)$. In order to determine the number of matrices A which fix the flag, we may therefore only consider the possibility $b_1 = 1$, $b_2 = b_3 = \ldots = b_{k-1} = 0$. Our conditions therefore reduce to $a_{11} = \gamma$; $a_{12} = a_{13} = \ldots = a_{1, k-1} = 0$; $a_{1k} + a_{kk}b_k = \gamma b_k^m$. There are $s - 1$ possibilities for each of the a_{11} and a_{kk}, s possibilities for each of $a_{21}, a_{31}, \ldots, a_{k-1, 1}, a_{2k}, a_{3k}, \ldots, a_{k-1, k}$ and $\Phi(s, k - 2)$ possibilities for

$$\begin{pmatrix} a_{22} & a_{23} & \cdots & a_{2,\,k-1} \\ a_{32} & a_{33} & \cdots & a_{3,\,k-1} \\ & & \vdots & \\ a_{k-1,\,2} & a_{k-1,\,3} & \cdots & a_{k-1,\,k-1} \end{pmatrix}.$$

Therefore, there are $(s-1)^2 s^{2k-4}\Phi(s, k-2)$ collineations which fix the flag.

c) Point $(c_1, c_2, \ldots, c_{k-1}, 0)$; Hyperplane $(0, 0, \ldots, 0, 1)$

$c_i \in GF(s)$;

$c_1, c_2, \ldots, c_{k-1}$ not all zero

$$A_1 \begin{pmatrix} c_1^m \\ c_2^m \\ \vdots \\ c_{k-1}^m \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_{k-1} \end{pmatrix}$$

The number of A_1 which satisfy this equation is constant for any choice of $c_1, c_2, \ldots, c_{k-1}$ not all zero. We may therefore choose $c_1 = 1, c_2 = c_3 = \ldots, c_{k-1} = 0$. Our conditions therefore reduce to $a_{11} = 1, a_{21} = a_{31} = \ldots = a_{k-1,\,1} = 0$. There are $(s-1)s^{2k-3}\Phi(s, k-2)$ collineations which fix the flag.

d) Point $(c_1, c_2, \ldots, c_{k-1}, 0)$; Hyperplane $(0, b_1, b_2, \ldots, b_k)$;

$c_i \in GF(s)$; $b_i \in GF(s)$;

$c_1, c_2, \ldots, c_{k-1}$ not all zero $b_1, b_2, \ldots, b_{k-1}$ not all zero;

$c_1b_1 + c_2b_2 + \ldots + c_{k-1}b_{k-1} = 0.$

$$A_1 \begin{pmatrix} c_1^m \\ c_2^m \\ \vdots \\ c_{k-1}^m \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_{k-1} \end{pmatrix}, \quad A_1' \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{k-1} \end{pmatrix} = \gamma \begin{pmatrix} b_1^m \\ b_2^m \\ \vdots \\ b_{k-1}^m \end{pmatrix}, \quad \gamma(\neq 0) \in GF(s)$$

$$a_{1k}b_1 + a_{2k}b_2 + \ldots + a_{kk}b_k = \gamma b_k^m$$

The number of A_1 satisfying the two matrix equations is constant for any values of $c_1, c_2, \ldots, c_{k-1}, b_1, b_2, \ldots, b_{k-1}$ subject to the restrictions of the values. This is true because the problem of determining the number of matrices A_1 which satisfy the two equations is equivalent to the problem of determining the number of elements in the stabilizer (in the group PC(k - 2, s)) of an incident point-hyperplane pair (flag) in a finite projective space PG(k - 2, s). In order to determine the number of matrices A which fix the flag, we therefore only need to consider the choice $c_1 = b_{k-1} = 1$, $c_2 = c_3 = \ldots = c_{k-1} = b_1 = b_2 = \ldots = b_{k-2} = 0$. Our restrictions therefore reduce to

$$a_{11} = 1;\ a_{k-1,\,k-1} = \gamma;\ a_{21} = a_{31} = \ldots = a_{k-1,\,1} = a_{k-1,\,2} = \ldots = a_{k-1,\,k-2} = 0;\ a_{k-1,\,k} + a_{kk}b_k^m = \gamma b_k^m$$

$$\gamma(\neq 0) \in GF(s).$$

For $k \geq 4$, there are (s - 1) possibilities for each of a_{kk} and $a_{k-1,\,k-1}$, s possibilities for each of $a_{1k}, a_{2k}, \ldots, a_{k-2,\,k}, a_{12}, a_{13}, \ldots, a_{1,\,k-1}, a_{2,\,k-2}, \ldots, a_{k-2,\,k-1}$ and $\Phi(s, k - 3)$

possibilities for

$$\begin{pmatrix} a_{22} & a_{23} & \cdots & a_{2,\,k-2} \\ a_{32} & a_{33} & \cdots & a_{3,\,k-2} \\ & & \vdots & \\ a_{k-2,\,2} & a_{k-2,\,3} & \cdots & a_{k-2,\,k-2} \end{pmatrix}.$$

Thus there are $(s-1)^2 s^{3k-7} \Phi(s, k-3)$ collineations which fix the flag. For $k = 3$, there are $(s-1)^2 s^2$ collineations which fix the flag.

e) Point $(c_1, c_2, \ldots, c_{k-1}, 0)$; Hyperplane $(1, b_1, b_2, \ldots, b_k)$;

$c_i \in GF(s)$; $b_i \in GF(s)$;

$c_1, c_2, \ldots, c_{k-1}$ not all zero $b_1, b_2, \ldots, b_{k-1}$ not all zero;

$1 + c_1 b_1 + c_2 b_2 + \ldots c_{k-1} b_{k-1} = 0$

$$A_1 \begin{pmatrix} c_1^m \\ c_2^m \\ \vdots \\ c_{k-1}^m \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_{k-1} \end{pmatrix}, \quad A_1' \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{k-1} \end{pmatrix} = \begin{pmatrix} b_1^m \\ b_2^m \\ \vdots \\ b_{k-1}^m \end{pmatrix}$$

$$a_{1k} b_1 + a_{2k} b_2 + \ldots + a_{kk} b_k = b_k^m$$

The number of A_1 satisfying the two matrix equations is constant for any values of $c_1, c_2, \ldots, c_{k-1}, b_1, b_2, \ldots, b_{k-1}$ satisfying $1 + c_1 b_1 + c_2 b_2 + \ldots + c_{k-1} b_{k-1} = 0$. This is true since the problem of determining the number of matrices A_1 which satisfy the equations is equivalent to the problem of determining the number of elements in

the stabilizer (in the group EC(k - 1, s)) of an incident point-hyperplane pair in a finite affine space EG(k - 1, s). We may therefore choose $c_1 = 1$, $b_1 = -1$, $c_2 = c_3 = \ldots = c_{k-1} = b_2 = b_3 = \ldots = b_{k-1} = 0$. For this choice, our restrictions reduce to

$$a_{11} = 1;\ a_{21} = a_{31} = \ldots\ a_{k-1,\,1} = a_{12} = a_{13} = \ldots = a_{1,\,k-1} = 0;$$

$$-a_{1k} + a_{kk}b_k = b_k^{\,m}.$$

There are $(s - 1)s^{k-2}\Phi(s, k - 2)$ collineations which fix the flag.

f) Point $(c_1, c_2, \ldots, c_k)$, $c_i \in GF(s)$, $c_k \neq 0$

Hyperplane $(0, b_1, b_2, \ldots, b_k)$; $b_i \in GF(s)$; $b_1, b_2, \ldots, b_{k-1}$ not all zero; $c_1b_1 + c_2b_2 + \ldots + c_kb_k = 0$

$$A_1 \begin{pmatrix} c_1^{\,m} \\ c_2^{\,m} \\ \cdot \\ \cdot \\ \cdot \\ c_{k-1}^{\,m} \end{pmatrix} + c_k^{\,m} \begin{pmatrix} a_{1k} \\ a_{2k} \\ \cdot \\ \cdot \\ \cdot \\ a_{k-1,\,k} \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ \cdot \\ \cdot \\ \cdot \\ c_{k-1} \end{pmatrix}, \quad A_1' \begin{pmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_{k-1} \end{pmatrix} = \gamma \begin{pmatrix} b_1^{\,m} \\ b_2^{\,m} \\ \cdot \\ \cdot \\ \cdot \\ b_{k-1}^{\,m} \end{pmatrix}$$

$$\gamma\ (\neq 0) \in GF(s)$$

$$a_{kk} = c_k^{\,1-m}$$

(The condition $a_{1k}b_1 + a_{2k}b_2 + \ldots + a_{kk}b_k = \gamma b_k^{\,m}$ is a linear combination of the other conditions.) There are $(s - 1)s^{k-2}\Phi(s, k - 2)$ possibilities for A_1'. When A_1' is determined, all of the other coefficients in A are completely determined. Thus, there are $(s - 1)s^{k-2}\Phi(s, k - 2)$ collineations which fix the flag.

g) Point $(c_1, c_2, \ldots, c_k)$, $c_i \in GF(s)$, $c_k \neq 0$ Hyperplane $(1, 0, 0, \ldots, 0, b)$, $b = -c_k^{-1}$

$$A_1 \begin{pmatrix} c_1^m \\ c_2^m \\ \cdot \\ \cdot \\ \cdot \\ c_{k-1}^m \end{pmatrix} + c_k^m \begin{pmatrix} a_{1k} \\ a_{2k} \\ \cdot \\ \cdot \\ \cdot \\ a_{k-1,\,k} \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ \cdot \\ \cdot \\ \cdot \\ c_{k-1} \end{pmatrix}, \quad a_{kk} = c_k^{1-m}$$

There are $\Phi(s, k-1)$ possibilities for A_1. Once A_1 is determined all of the other coefficients of A are completely determined. Thus, there are $\Phi(s, k-1)$ collineations which fix the flag.

h) Point $(c_1, c_2, \ldots, c_k)$; $c_i \in GF(s)$; $c_k \neq 0$ Hyperplane $(1, b_1, b_2, \ldots, b_k)$; $b_i \in GF(s)$; $b_1, b_2, \ldots, b_{k-1}$ not all zero; $1 + c_1b_1 + c_2b_2 + \ldots + c_kb_k = 0$

$$A_1 \begin{pmatrix} c_1^m \\ c_2^m \\ \cdot \\ \cdot \\ \cdot \\ c_{k-1}^m \end{pmatrix} + c_k^m \begin{pmatrix} a_{1k} \\ a_{2k} \\ \cdot \\ \cdot \\ \cdot \\ a_{k-1,\,k} \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ \cdot \\ \cdot \\ \cdot \\ c_{k-1} \end{pmatrix}, \quad A_1' \begin{pmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_{k-1} \end{pmatrix} = \begin{pmatrix} b_1^m \\ b_2^m \\ \cdot \\ \cdot \\ \cdot \\ b_{k-1}^m \end{pmatrix}, \quad a_{kk} = c_k^{1-m}$$

There are $s^{k-2}\Phi(s, k-2)$ possibilities for A_1'. Once A_1' is determined, all of the other coefficients of A are completely determined. Thus, there are $s^{k-2}\Phi(s, k-2)$ collineations which fix the flag.

APPENDIX F

We assume a random blocks model ($\lambda = 1$); thus for $P = E_3 + E_4$,

$$N = I - P = -\frac{1}{\phi}(E_0 + E_1) + \left(\frac{\phi + 1}{\phi}\right) E_2 + \frac{1}{\phi} E_3 + E_5 + E_6, V_1 = E_0 + E_1,$$

$V_2 = E_2$, $V_3 = E_3$, $V_4 = E_4$, $V_5 = E_5$ and $V_6 = E_6$,

$$NV_1N = 0$$
$$NV_2N = -\frac{1}{\phi + 1}(E_0 + E_1) + E_2 + \left(\frac{1}{\phi + 1}\right) E_3$$
$$NV_3N = 0$$
$$NV_4N = 0$$
$$NV_5N = E_5$$
$$NV_6N = E_6 .$$

The equations (6.3.1) thus reduce to

$$\operatorname{tr}\left(\frac{\phi}{\phi + 1}\right)\left[-\left(\frac{1}{\phi + 1}\right)(E_0 + E_1) + E_2 + \left(\frac{1}{\phi + 1}\right) E_3\right] \alpha_2 = \nu_2$$

$$\operatorname{tr} E_5 \ \alpha_5 = \nu_5$$

$$\operatorname{tr} E_6 \ \alpha_6 = \nu_6 .$$

Since $\operatorname{tr} E_0 = \operatorname{tr} E_1 = \operatorname{tr} E_2 = \operatorname{tr} E_3 = v - 1$, $\operatorname{tr} E_5 = n - b - v + 1$, and $\operatorname{tr} E_6 = b - v$ (Sysoev and Shaikin (1976, III)), we have

$$\frac{\phi}{\phi + 1}\left[-\left(\frac{1}{\phi + 1}\right)(v - 1) + v - 1\right] \alpha_2 = \nu_2 \rightarrow \frac{\phi^2}{(\phi + 1)^2}(v-1)\alpha_2 = \nu_2$$
$$(n - b - v + 1) \ \alpha_5 = \nu_5$$
$$(b - v) \ \alpha_6 = \nu_6 .$$