ABSTRACT

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Vertex algebras [Bor86] are important algebraic structures that occupy a prominent place in the intersection between mathematics and physics. On the one hand, they provide the basis for a rigorous algebraic approach to (two-dimensional) conformal field theory (CFT) [BPZ84; FMS12], while on the other hand, vertex algebras and their twisted modules [Don94; FLM89; see also Bak16] played a prominent role in the development (by Richard Borcherds [Bor92a]) of tools used to settle the famous Conway–Norton [CN79] conjectures regarding the moonshine module [see also LL04]. Since their beginnings, the theory around vertex algebras has expanded in many different directions, spurred on not only by purely mathematical interests, but also by developments in physics related to quantum field theory.

One such direction has been parallel to the demands of logarithmic conformal field theory (LCFT) in physics [Gur93; Gur13; CR13], and this is the setting of the current investigations. Recently, Bakalov and Villarreal [BV22] published a framework for describing so-called *logarithmic* vertex algebras (logVAs), which allow for logarithms of the formal variable in the fields defined by the algebra. This framework is general enough to allow for the rigorous development of LCFT, and allows for computation with the operator product expansion, and hence is valuable to the physicist. Further, a more recent paper by the same authors [BV23] established a close mathematical relationship between logarithmic vertex algebras and non-local Poisson vertex algebras (non-local PVAs) [DSK13], which themselves have close connections to integrable systems.

The present work develops a theorem on the existence of logVAs constructed from a set of generators (and of particular interest, a set of generators of a finite-dimensional Lie algebra) under certain relatively mild conditions, and gives several examples of such logVAs. A PBW-type theorem is given for the logVAs so constructed which has potential application beyond

the logVAs constructed herein. Additionally, an existence theorem based on work in [BV23] establishes new examples of non-local PVAs, which have applications to the theory of integrable systems.

Advances in Logarithmic Vertex Algebras

by
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DEDICATION

To my family.

BIOGRAPHY

The author was born in Stockholm, Sweden and spent many formative years (and summers) there. Settling permanently in the United States at the age of 10, he grew up mainly in Louisville, Kentucky, eventually ending up in North Carolina, with the ellipsis partially filled by sojourns in New York and Europe.

In his spare time, he enjoys spending time with his family, reading, and playing music, chess and various sports.

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CHAPTER

1

INTRODUCTION

Vertex algebras [Bor86] are important algebraic structures that arose out of the vertex-operator realizations of affine Kac-Moody algebras [Kac07; Moo68; BP02], and relatedly, during the search for a resolution to the Conway-Norton conjectures relating to the moonshine module [Bor92b; FLM89]. Additionally, vertex algebras occupy a prominent place in the intersection between mathematics and physics, since they provide the basis for a rigorous algebraic approach to (two-dimensional) conformal field theory (CFT) [BPZ84; FMS12], while on the other hand, vertex algebras and their twisted modules [Don94; FLM89; Hua10; see also Bak16] played a prominent role in the development by Richard Borcherds [Bor92b] of tools used to settle the aforementioned Conway–Norton [Bor92a] conjectures regarding the moonshine module [see also LL04]. The language of vertex algebras has by now become standard in descriptions of conformal field theory, and subsequently, the interests of theoretical developments in physics have given rise to avenues of research in the theory around vertex algebras. The theory around vertex algebras has expanded in many different directions, spurred on not only by developments in physics related to quantum field theory, but also by purely mathematical interests.

One direction of this expansion of the theory of vertex algebras with roots in *both* mathematics and physics is the development, due to Bakalov and Villarreal [BV22], of *logarithmic vertex algebras*, which generalize the notion of a vertex algebra by allowing for logarithms of the formal variables in the operator expansions (fields). This generalization has been parallel to the demands of logarithmic conformal field theory (LCFT) in physics [CR13; Gur93; GL05; Gur13], and this is the setting of the present work. The theory of [BV22] is general enough to allow for the rigorous development of LCFT, and allows for computation with the operator product expansion, and hence is valuable to the physicist. Further, a more recent paper by the same authors [BV23] established a close mathematical relationship between logarithmic vertex algebras and non-local Poisson vertex algebras (non-local PVAs) [DSK13; DSKVW19], which themselves have close connections to integrable systems. It should be noted that the notion of a logarithmic vertex algebras as defined in [BV22] and from which we proceed is not the same as the related notion of a logarithmic intertwining operator, which is discussed in e.g. [HLZ14; Mil02; Mil07], as the logarithmic singularities are present in the algebra itself, and not only in the intertwining operators; see [BV22] for further discussion of this difference and the connection between these two notions.

On the mathematical side, the notion of a logarithmic vertex algebra has connections to twisted modules of vertex algebras [Don94; FLM89]. In the theory of such objects, one may generalize by allowing logarithms of formal variables in the quantum fields, thus obtaining *twisted logarithmic modules* of vertex algebras [Bak16; Hua10]. A logarithmic vertex algebra is then an extension of the structure of a twisted logarithmic module in the sense that it allows logarithmic singularities in the algebra itself, and not merely in its action on modules.

In slightly finer detail, a logarithmic vertex algebra [BV22] is a structure on a vector superspace V given in terms of a so-called (infinitesimal) *braiding map* $\mathscr{S} \in \mathrm{End}(V) \otimes \mathrm{End}(V)$

which is locally nilpotent in the sense that for every pair $a, b \in V$, $\mathscr{S}^r(a \otimes b) = 0$ for some positive integer r. The properties of a logarithmic vertex algebra may be consequently be described in much the same way as those of ordinary vertex algebras, and the latter are simply logarithmic vertex algebras for which $\mathscr{S} = 0$. Indeed, just as for ordinary vertex algebras, we have a state-field correspondence (linear map) associating to every vector a in a logVA V a field $Y(a,z)$, but now the field $Y(a,z)$ is an element of the space $V(\!(z)\!)[\zeta]$, where $\zeta = \log z$ is a new formal variable. The fields $Y(a,z)$ may still be expanded in a familiar form in terms of modes, which now crucially involve the braiding map. For $a, b \in V$, we have

$$Y(a,z)b = \sum_{n \in \mathbb{Z}} z^{-n-1-\mathscr{S}} a_{(n+\mathscr{S})} b$$

where $z^{-\mathscr{S}} = e^{-\zeta \mathscr{S}}$, and the expression $a_{(n+\mathscr{S})}b$ is an analog of the n-th product in an ordinary VA. In slightly more detail, we have bilinear products $\mu_{(n)} : V \otimes V \to V$, and $a_{(n+\mathscr{S})}b = \mu_{(n)}(a \otimes b)$, whence symbolically we have

$$\begin{aligned} Y(a,z)b &= \sum_{n \in \mathbb{Z}} \mu_{(n)}\big(z^{-n-1-\mathscr{S}}(a \otimes b)\big) \\ &= \sum_{\substack{i \geq 0 \\ n \in \mathbb{Z}}} \frac{(-1)^i}{i!} \zeta^i z^{-n-1} \mu_{(n)}\big(\mathscr{S}^i(a \otimes b)\big), \end{aligned}$$

and from which the logarithmic quality of the fields is apparent.

Logarithmic vertex algebras satisfy a version of the Borcherds identity from the theory of ordinary vertex algebras, and this identity plays a fundamental role in the present work. Here, we develop a result that allows the construction of a new class of examples of logVAs in a natural way. More specifically, we enumerate conditions which a logVA generated under the nth modes of its subspace U must satisfy under certain assumptions on its nonnegative nth products, and we use these conditions to construct a logVA V satisfying these conditions by realizing it as a module over an associative algebra satisfying relations determined from the Borcherds identity in logVAs.

Explicitly, we have the following result:

Theorem. *Let U be a superspace endowed with an even skew super symmetric bilinear product $[\![\cdot,\cdot]\!] : U\times U \to U$ and an even super symmetric bilinear product $(\cdot|\cdot) : U\times U \to \mathbb{C}$. Let $L \geq 1$ be a positive integer, and let ϕ_i, ψ_i be nilpotent endomorphisms of U for $1 \leq i \leq L$ which satisfy $[\phi_i,\phi_j] = [\phi_i,\psi_j] = [\psi_i,\psi_j] = 0$ for $1 \leq i, j \leq L$, and suppose further that all ϕ_i,ψ_i are derivations of the products $[\![\cdot,\cdot]\!]$ and $(\cdot|\cdot)$. Suppose also that $(\cdot|\cdot)$ is invariant with respect to $[\![\cdot,\cdot]\!]$. Let $\mathscr{S} \in End(U\otimes U)$ be a braiding map on U given by (2.31), and suppose $\mathscr{S}$ is locally nilpotent. Suppose also that the following modified Jacobi identity holds for all $a, b, c \in U$:*

$$[\![a,[\![b,c]\!]]\!] - (-1)^{p(a)p(b)}[\![b,[\![a,c]\!]]\!] - [\![[\![a,b]\!],c]\!] = \tag{1.1}$$
$$\sum_{\ell=1}^{L}(-1)^{p(\psi_\ell)\left(p(a)+p(b)\right)}\Big(\big(\phi_\ell a\big|\, b\big)\psi_\ell c + (-1)^{p(\psi_\ell)p(b)}\big(\psi_\ell b\big|\, c\big)\phi_\ell a - (-1)^{p(a)p(b)+p(\psi_\ell)p(a)}\big(\psi_\ell a\big|\, c\big)\phi_\ell b\Big).$$

Then there exists a logVA V containing U whose braiding map coincides with $\mathscr{S}$ on $U\otimes U$, and in which the nonnegative nth products $\mu_{(n)} := \mu_{(n+\mathscr{S})}$ restricted to $U\otimes U$ are given by

$$\mu_{(n)}(a\otimes b) = \begin{cases} [\![a,b]\!], & n=0,\\ (a\,|\,b)\,\mathbb{1}, & n=1,\\ 0, & n\geq 2.\end{cases} \tag{1.2}$$

Additionally, we show that the so-constructed logVA is nontrivial by proving a PBW-type theorem, Theorem 21 for it. This PBW Theorem may apply more broadly to other logVAs constructed prior to this work.

The logVA constructed as in the preceding theorem may be thought of as a generalization of the notion of a universal affine vertex algebra $V^\kappa(\mathfrak{g})$ [Kac17]. Indeed, in the case $\mathscr{S} = 0$, the logVA V constructed reduces to the universal affine vertex algebra $V^\kappa(\mathfrak{g})$—see Example 7 and Remark 18 below—where $\mathfrak{g}$ is a Lie algebra generated by U with Lie bracket given by $[\![\cdot,\cdot]\!]$ and bilinear form given by $(\cdot|\cdot)$.

We also obtain, as a consequence of the theorems here and a result in [BV23] on the existence of a non-local *Poisson vertex algebra* (non-local PVA) [DSK13] structure on the associated graded of a filtered logVA, the existence of non-local PVAs derived from the logVAs constructed, where a non-local PVA is a generalization of a PVA in which the lambda bracket is given by a formal Laurent series in λ^{-1} [BV23; DSK13]. An explicit description of the λ-bracket is obtained in our cases.

After proving these main results, we apply our theorem to obtain novel examples of logVAs, based both on known Lie superalgebras, and also more exotic algebras obtained through computational search. In particular, we obtain logVAs arising from $\mathfrak{sl}(2)$ and $\mathfrak{gl}(1|1)$, as well as logVAs arising from structures satisfying a modified version of the Jacobi identity for Lie algebras, (1.1), which holds for all $a, b, c \in U$, the generating subspace of the constructed logVA.

The examples arising from Lie algebras have the double bracket $[\![\cdot,\cdot]\!]$ equal to the Lie bracket, and so the left-hand side of the above identity is trivially satisfied. On the other hand, we also obtain algebras that are *not* Lie algebras with respect to the double bracket.

From the present work, a number of questions for further research present themselves, and some of these are discussed in the final chapter. In particular, it would be satisfying to have a sort of "reverse" construction of a logVA from a given non-local PVA (or non-local Lie conformal algebra). The generalization and application of our PBW Theorem to other logVAs is also anticipated in future work.

CHAPTER

2

PRELIMINARIES

In this section, some preliminary definitions and notions needed throughout the work are established, along with the notational conventions to be used with them. We will work over the complex number field, $\mathbb{C}$, unless otherwise stated. References for this section include [Kac17; Bak16; BS19; BV22].

2.1 Lie Algebras and Lie Superalgebras

2.1.1 Superspaces, Superalgebras, and Lie Superalgebras

We begin with the notion of a *vector superspace*, and then define in turn a *Lie superalgebra* and the special case of a *Lie algebra*, the latter two of which will arise in several of the main examples and results in later chapters.

Definition 1 (Vector Superspace)**.** A *vector superspace* (or simply *superspace*) is a vector space V with a $\mathbb{Z}_2$-grading. In other words, V has a natural direct-sum decomposition $V = V_{\bar{0}} \oplus V_{\bar{1}}$.

In the definition of a superalgebra, elements of $V_{\bar{0}}$ known as *even* elements, and those of $V_{\bar{1}}$ known as *odd* elements. Generally, elements of either $V_{\bar{0}}$ or $V_{\bar{1}}$ are known as *homogeneous elements*. For any homogeneous element $v \in V$, we define its *parity*, $p(v)$, by

$$p(v) = \begin{cases} 0, & \text{if } v \in V_{\bar{0}}, \\ 1, & \text{if } v \in V_{\bar{1}}. \end{cases} \tag{2.1}$$

Remark 1. A superspace $V = V_{\bar{0}} \oplus V_{\bar{1}}$ such that $V = V_{\bar{0}}$ (or equivalently, such that $V_{\bar{1}} = 0$) in which all elements are even is simply a vector space.

In the sequel, we will need to construct tensor products of superspaces. Let now $V = V_{\bar{0}} \oplus V_{\bar{1}}$ and $W = W_{\bar{0}} \oplus W_{\bar{1}}$ be two vector superspaces. Then the tensor product $V \otimes W$ is the superspace with components defined by

$$(V \otimes W)_\alpha = \bigoplus_{\beta \in \mathbb{Z}_2} V_\beta \otimes W_{\alpha-\beta}, \qquad \alpha \in \mathbb{Z}_2. \tag{2.2}$$

Starting from a vector superspace, one may define the notion of a *superalgebra*.

Definition 2 (Superalgebra)**.** Given a superspace A, a *superalgebra* structure on A is a bilinear multiplication $\mu : A \times A \to A$ on A, denoted here for simplicity by juxtaposition (so that $ab := \mu(a, b)$), which satisfies

$$ab \in A_{\alpha+\beta} \quad \text{whenever} \quad a \in A_\alpha,\ b \in A_\beta, \quad \alpha, \beta \in \mathbb{Z}_2. \tag{2.3}$$

The superalgebra A is called *associative* if its multiplication is associative; that is, if for any $a, b, c \in A$, we have $a(bc) = (ab)c$.

Starting from a vector superspace, one may define the notion of a *Lie superalgebra*, which is the extension of the notion of a Lie algebra to the setting of superspaces. It is also a case of a superalgebra, but is in general not associative.

Definition 3 (Lie Superalgebra)**.** A *Lie superalgebra* is a vector superspace V endowed with a superbilinear product $[\cdot,\cdot] : V \times V \to V$ (called the *Lie superbracket on* V) which satisfies the following identities.

(i) (*Super skew-symmetry*) $\quad [x, y] = -(-1)^{p(x)p(y)}[y, x] \qquad x, y \in V$

(ii) (*Super Jacobi identity*)

$$(-1)^{p(x)p(z)}[x,[y,z]]+(-1)^{p(y)p(x)}[y,[z,x]]+(-1)^{p(z)p(y)}[z,[x,y]]=0, \qquad x,y,z \in V. \tag{2.4}$$

Remark 2. It is important to note that the defining identities above are written for homogeneous elements $x, y, z \in V$.

Remark 3. In the case that V is a usual vector space ($V = V_{\bar{0}}$), the resulting structure is simply a *Lie algebra.*

Remark 4. Super skew-symmetry implies that for any even element $x \in V_{\bar{0}}$, $[x, x] = 0$, while for any odd element $y \in V_{\bar{1}}$, $[[y, y], y] = 0$.

Let us illustrate these definitions via some pertinent examples.

Example 1. Let p and q be nonnegative integers. The superspace $V = \mathbb{C}^{p|q}$ is the vector superspace $\mathbb{C}^{p+q}$ with $V_{\bar{0}} = \mathbb{C}^p$ and $V_{\bar{1}} = \mathbb{C}^q$. (This may be generalized to $\mathbb{F}^{p|q}$, replacing $\mathbb{C}$ by any field $\mathbb{F}$.) The *superdimension* of $\mathbb{C}^{p|q}$ is often denoted by the symbol $p|q$.

Example 2. Suppose that A is an *associative superalgebra*, with multiplication denoted by juxtaposition. Then the *commutator bracket* on A is a bilinear product, often denoted by $[\cdot,\cdot] : A \times A \to A$, which is defined for homogeneous elements $a, b \in A$ by

$$[a, b] = ab - (-1)^{p(a)p(b)} ba, \tag{2.5}$$

and extended to all of A bilinearly. It is a simple matter to check that the commutator bracket endows A with the structure of a Lie superalgebra.

Example 3. Builiding upon the previous example, a simple but important example of a Lie superalgebra is furnished by the endomorphism ring $L = \text{End}(V)$ of endomorphisms of a vector superspace V. The superspace structure on L is defined in a natural way by

$$L_\alpha = \left\{\varphi \in L \mid \varphi(V_\beta) \subset V_{\alpha+\beta},\ \beta \in \mathbb{Z}_2\right\}, \qquad \alpha \in \mathbb{Z}_2. \tag{2.6}$$

Then, the commutator bracket on L provides it with the structure of a Lie superalgebra.

2.1.2 Tensor Algebras, Universal Enveloping Algebras, and the PBW Theorem

In this subsection, we give a very brief overview of the construction of the universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ (the case of a Lie super algebra is analogous), and state the Poincaré-Birkhoff-Witt (PBW) Theorem for $U(\mathfrak{g})$, since we will perform a similar construction in Chapter 3. Refer to [Hum12] for further details.

Let $\mathcal{T} = \bigoplus_{m\geq 0} \mathcal{T}^m$ be the tensor algebra over $\mathfrak{g}$, where $\mathcal{T}^0 = \mathbb{C}$ and $\mathcal{T}^m = \mathfrak{g}^{\otimes m}$ is the space of homogeneous tensors of degree m for $m \geq 1$. $\mathcal{T}$ is the free unital associative algebra over $\mathfrak{g}$ under the tensor product, and is generated by its subspace $\mathfrak{g}$.

The *universal enveloping algebra* $U(\mathfrak{g})$ is the quotient $U(\mathfrak{g}) = \mathcal{T}/\mathcal{I}$ of $\mathcal{T}$ by the two-sided ideal

$$\mathcal{I} = \text{span}\left\{t \otimes \left(g \otimes h - h \otimes g - [g,h]\right) \otimes u \mid t, u \in \mathcal{T},\ g, h \in \mathfrak{g}\right\}. \tag{2.7}$$

As such, it is again a unital associative algebra, endowed with a product descended from $\mathcal{T}$, given simply by juxtaposition. Since $\mathcal{I}$ is not a homogeneous ideal, $U(\mathfrak{g})$ is not graded in general, but it is a filtered algebra.

A fundamental result for $U(\mathfrak{g})$ is the following well-known theorem, due to Poincaré, Birkhoff and Witt. It gives an explicit description of a basis for the universal enveloping algebra in terms of an ordered basis for $\mathfrak{g}$.

Theorem 5 (PBW Theorem for Lie algebras)**.** Let $\mathfrak{g}$ be a Lie algebra with basis $\{g_i \mid i \in I\}$, where I is a totally ordered set with order $\leq$. Then the set of ordered monomials

$$\{g_{i_1} \cdots g_{i_m} \mid i_1 \leq \cdots \leq i_m,\ i_s \in I\} \tag{2.8}$$

is a basis for $U(\mathfrak{g})$.

2.1.3 Examples of Lie Superalgebras

In this subsection, we will briefly present some examples of Lie (super)algebras. The notation for each algebra will be revived without comment in the sequel. See e.g. [Kac17] for further details.

Example 4 (General Lie superalgebra)**.** Let m, n be positive integers. Then the *general Lie superalgebra* $\mathfrak{gl}(m|n)$ is the associative superalgebra $\mathrm{End}(\mathbb{C}^{m|n})$ endowed with the commutator bracket. Let us single out two sub-examples of this construction.

Example 5 (General and Special Lie algebras)**.** Let n be a positive integer. The Lie superalgebra $\mathfrak{gl}(n|0) =: \mathfrak{gl}(n)$ is called the *general Lie algebra.* It is realized as the set of all $n \times n$ matrices over $\mathbb{C}$. $\mathfrak{gl}(n)$ has a Lie subalgebra $\mathfrak{sl}(n)$ consisting of all the trace-zero $n \times n$ matrices over $\mathbb{C}$, called the *special linear Lie algebra.* In this work, the *standard basis for* $\mathfrak{sl}(n)$ will be the set

$$\mathcal{B}_n = \{E_{1,1} - E_{2,2}, E_{2,2} - E_{3,3}, \ldots, E_{n-1,n-1} - E_{n,n}\} \cup \{E_{i,j} \mid 1 \leq i, j \leq n, i \neq j\}, \tag{2.9}$$

where $\{E_{i,j}\}$ is the standard basis for the $n \times n$ matrices over $\mathbb{C}$. Let us present here the commutation relations for $\mathfrak{sl}(2)$. Let

$$\mathcal{B}_2 = \{h = E_{1,1} - E_{2,2},\ e = E_{1,2},\ f = E_{2,1}\}. \tag{2.10}$$

Then we have

$$[h,e]=2e,\ [h,f]=-2f,\ [e,f]=h, \tag{2.11}$$

with the remaining brackets deduced from skew-symmetry.

Example 6. The superalgebra $\mathfrak{gl}(1|1)$ is the Lie superalgebra $\mathfrak{g}=\mathfrak{g}_{\bar{0}}\oplus\mathfrak{g}_{\bar{1}}$ where

$$\mathfrak{g}_{\bar{0}}=\operatorname{span}\left\{N=\begin{bmatrix}1/2 & 0\\ 0 & -1/2\end{bmatrix},\ E=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right\} \tag{2.12}$$

and

$$\mathfrak{g}_{\bar{1}}=\operatorname{span}\left\{\psi^{+}=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix},\ \psi^{-}=\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}\right\}. \tag{2.13}$$

The nonzero superbrackets in $\mathfrak{g}$ are then

$$[N,\psi^{\pm}]=\pm\psi^{\pm},\quad [\psi^{+},\psi^{-}]=E, \tag{2.14}$$

with the rest obtained from these by super skew-symmetry. Note that E is central in $\mathfrak{g}$.

2.2 Vertex Algebras and Logarithmic Vertex Algebras

In this section, we define the main objects of consideration in this work, namely, logarithmic vertex algebras. In order to motivate the definition of logarithmic vertex algebras, it is advantageous first to recall the definition of an ordinary vertex algebra.

2.2.1 Vertex Algebras

Vertex algebras [Bor86] (which we will at times shorten to VAs) are important algebraic structures that occupy a prominent place in the intersection between mathematics and physics. On the one hand, they provide the basis for a rigorous algebraic approach to (two-dimensional) conformal field theory (CFT) [BPZ84; FMS12], while on the other hand, vertex algebras and their twisted modules [Don94; FLM89; Hua10; Bak16] played a prominent role in the development [Bor92b] of tools used to settle the famous Conway–Norton [CN] conjectures regarding the moonshine module.

Although vertex algebras may be defined in a number of equivalent ways, the best definition for our purposes is the one given in [BV22]. However, before giving the definition of a vertex algebra, a few further definitions need to be presented, which will again be generalized once we are prepared to define logarithmic vertex algebras.

Definition 4 (Formal Distribution, Quantum Field)**.** Let V be a superspace. A *formal distribution on V* is an element of $V[[z,z^{-1}]]$, the space of formal power series with coefficients in V. In symbols, if $a \in V[[z,z^{-1}]]$, then we write

$$a = a(z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}, \qquad a_{(n)} \in V. \tag{2.15}$$

Then a *quantum field on V* (or simply, *field*) is a formal distribution with coefficients in $\mathrm{End}(V)$, subject to the condition that for every $v \in V$,

$$a_{(n)} v = 0, \qquad n \gg 0, \tag{2.16}$$

that is, $a_{(n)} v = 0$ for n sufficiently large.

Quantum fields are a convenient structure for assembling and manipulating a collection of endomorphisms of a superspace. Note that set of quantum fields is a vector space under the natural coefficient-wise scalar multiplication and addition, but it is *not* in general an algebra under the usual multiplication. Instead, in order to guarantee the convergence of each coefficient, and to guarantee that the resulting product is again a field, we must introduce a so-called *normal ordering* of the terms of a product of fields. This is done by partitioning a field $a(z)$ into a sum of two parts, $a(z) = a(z)_+ + a(z)_-$, where

$$a(z)_+ = \sum_{n<0} a_{(n)} z^{-n-1} \quad \text{and} \quad a(z)_- = \sum_{n\geq 0} a_{(n)} z^{-n-1}. \tag{2.17}$$

Then, given two fields $a(z)$ and $b(z)$, their *normally-ordered product* is the field $:a(z)b(z):$ given by

$$:a(z)b(z):= a(z)_+ b(z) + (-1)^{p(a)p(b)} b(z)a(z)_-, \tag{2.18}$$

and where in the above, products are taken in the usual way for series. We will denote by $\mathrm{Fie}(V)$ the space of (quantum) fields on a superspace V.

The setting for defining vertex algebras is now in place.

Definition 5 (Vertex Algebra)**.** A *vertex algebra* V is a vector superspace endowed with a distinguished even vector $\mathbb{1} \in V_{\bar{0}}$ (the *vacuum vector*), an even endomorphism T (the *translation operator*), and an even linear map (the *state-field correspondence*) $Y : V \to \mathrm{Fie}(V)$ given by

$$a \mapsto Y(a,z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}, \quad a_{(n)} \in \mathrm{End}(V) \tag{2.19}$$

such that for every $a, b \in V$, the following axioms hold.

$$(\textbf{translation covariance}): [T, Y(a,z)] = \partial_z Y(a,z), \tag{2.20}$$

$$(\textbf{vacuum}): T\mathbb{1} = 0, \quad Y(\mathbb{1}, z) = I, \quad Y(a,z)|_{z=0} = a, \tag{2.21}$$

$$(\textbf{locality}): (z-w)^N [Y(a,z), Y(b,w)] = 0 \quad \text{for} \quad N \gg 0. \tag{2.22}$$

The endomorphisms $a_{(n)}$ above are called the *modes* of a, and they satisfy the following *Borcherds identity* for $a, b, c \in V$ and $n, m, k \in \mathbb{Z}$:

$$\sum_{i\geq 0} (-1)^i \binom{n}{i} \Big(a_{(m+n-i)}(b_{(k+i)}c) - (-1)^{n+p(a)p(b)} b_{(k+n-i)}(a_{(m+i)}c) \Big)$$

$$= \sum_{j\geq 0} \binom{m}{j} (a_{(n+j)}b)_{(m+k-j)}c. \tag{2.23}$$

The Borcherds identity [Bor86; Kac17] is a very useful set of identities, which establishes many of the properties satisfied by vertex algebras. It is the analogue of this identity in the more general setting of logarithmic vertex algebras which will play a decisive role in establishing not only the main results in this work, but also the main examples of logarithmic vertex algebras.

2.2.2 Examples of Vertex Algebras

Let us demonstrate the definition of a vertex algebra via two of the principal examples of the structure, namely, the Heisenberg VA and the universal affine VA. The former is in fact a special case of the latter, while the latter will be a particularly useful example for us, since it is closely related to the examples we construct in our Existence Theorem. We start by presenting the universal affine VA.

Example 7 (Universal affine VA)**.** Let $\mathfrak{g}$ be a finite-dimensional Lie algebra, and let $(\cdot|\cdot)$ be a nondegenerate symmetric invariant (with respect to the Lie bracket) bilinear form on it. First, define the *affine Lie algebra* $\hat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K$ to have the Lie brackets given by

$$\begin{aligned} [a t^m, b t^n] &= [a, b] t^{m+n} + m\delta_{m,-n}(a\,|\,b)K, \qquad (2.24)\\ [K, \hat{\mathfrak{g}}] &= 0. \end{aligned}$$

For each $\kappa \in \mathbb{C}$, called the *level*, define the Verma module $M(\kappa\Lambda_0)$ by

$$M(\kappa\Lambda_0) = \operatorname{Ind}^{\hat{\mathfrak{g}}}_{\mathfrak{g}[t]\oplus\mathbb{C}K}\mathbb{C} = U(\hat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t]\oplus\mathbb{C}K)} \mathbb{C}, \tag{2.25}$$

where we let $\mathfrak{g}[t]$ act trivially on $\mathbb{C}$, and let K act as κ. $M(\kappa\Lambda_0)$ is then a $\hat{\mathfrak{g}}$-module of highest weight, with highest weight vector denoted $\mathbb{1}$ (the image of 1).

It can be shown [FZ92] that $M(\kappa\Lambda_0)$ has the structure of a VA, which is usually denoted $V^\kappa(\mathfrak{g})$, called the *universal affine vertex algebra at level* κ. The vacuum vector is $\mathbb{1}$, and we identify $a \in \mathfrak{g}$ with $(a\,t^{-1})\mathbb{1} \in V^{\kappa}(\mathfrak{g})$, so that the modes are denoted by $a_{(m)} = a\,t^{m}$. It is generated as a VA by the local fields

$$Y(a,z) = \sum_{m\in\mathbb{Z}} a_{(m)} z^{-m-1}, \tag{2.26}$$

and the commutator formula implies that the Lie brackets are equivalent to the relations $(a, b \in \mathfrak{g})$

$$a_{(0)}b = [a,b], \quad a_{(1)}b = (a\,|\,b)\kappa\mathbb{1}, \quad a_{(m)}b = 0,\ m \geq 2. \tag{2.27}$$

See e.g. [Kac17] for further details.

Although it is a special occurrence of the universal affine VA, the Heisenberg VA is so well-known as an example that it deserves explicit mention.

Example 8 (Heisenberg VA)**.** Take $\mathfrak{g} = \mathfrak{h}$ to be an abelian Lie algebra in the previous example, $[\mathfrak{h},\mathfrak{h}] = 0$. Then the Heisenberg VA (at level κ) is $V^{\kappa}(\mathfrak{h})$. It turns out that $V^{\kappa}(\mathfrak{h}) \cong V^{1}(\mathfrak{h})$ for any $\kappa \neq 0$, so we can take $\kappa = 1$. The resulting VA is usually denoted $\mathcal{F}$, and is also known as the *(bosonic) Fock space.*

2.2.3 Logarithmic Vertex Algebras

The definition of logarithmic vertex algebras [BV22] (logVAs at times from here on) can be said to have arisen out of the lack of a precise mathematical formalism for logarithmic conformal field theory (LCFT). While in the physics literature, CFTs with logarithmic singularities have been the object of investigation for a long time by now, a rigorous mathematical treatment was until recently lacking. Bakalov and Villarreal have given a definition of logVAs which we follow here; see [BV22] for full details; the presentation here follows theirs quite closely. A few preliminary definitions mirroring those for VAs need to be made before the main definition can be given.

Logarithmic Fields and Braiding Maps

Here and throughout, let V be a vector superspace, let z and ζ (and any indexed versions thereof) be commuting formal even variables, where ζ is to be treated as $\zeta = \log z$. Define also the notation $z_{ij} = z_i - z_j$, and similarly for other variables. Let $D_z = \partial_z + z^{-1}\partial_\zeta$, and define the formal power series

$$\vartheta_{12} = \zeta_1 - \sum_{n\geq 1} \frac{1}{n} z_1^{-n} z_2^{n} \qquad \text{and} \qquad \vartheta_{21} = \zeta_2 - \sum_{n\geq 1} \frac{1}{n} z_1^{n} z_2^{-n}. \tag{2.28}$$

We will also need the following series expansions. Define the series expansions of z_{12}^{-1} and of $\zeta_{12} = \log z_{12}$ in the domain $|z_1| > |z_2|$ by:

$$\iota_{z_1,z_2} z_{12}^{-1} = D_{z_1} \vartheta_{12} = \sum_{j\geq 0} z_1^{-j-1} z_2^{j}, \tag{2.29a}$$

$$\iota_{z_1,z_2} \zeta_{12} = \vartheta_{12} = \zeta_1 - \sum_{j\geq 1} \frac{1}{j} z_1^{-j} z_2^{j}, \tag{2.29b}$$

and observe that by defining

$$\iota_{z_1,z_2}\left(z_i^{\pm 1}\right) = z_i^{\pm 1},$$

$$\iota_{z_1,z_2}(\zeta_i) = \zeta_i$$

for $i = 1, 2$, we can extend ι_{z_1,z_2} uniquely to a homomorphism of associative algebras

$$\iota_{z_1,z_2} : V[\![z_1, z_2]\!][z_1^{-1}, z_2^{-1}, z_{12}^{-1}, \zeta_1, \zeta_2, \zeta_{12}] \to V(\!(z_1)\!)(\!(z_2)\!)[\zeta_1, \zeta_2]. \tag{2.29c}$$

Define—*mutatis mutandis*—the homomorphism

$$\iota_{z_2,z_1} : V[\![z_1, z_2]\!][z_1^{-1}, z_2^{-1}, z_{12}^{-1}, \zeta_1, \zeta_2, \zeta_{12}] \to V(\!(z_2)\!)(\!(z_1)\!)[\zeta_1, \zeta_2], \tag{2.29d}$$

which is expansion in the domain $|z_2| > |z_1|$.

The space of logarithmic fields generalizing $\mathrm{Fie}(V)$ is defined in terms of z and ζ.

Definition 6 (Logarithmic Quantum Fields)**.** Given a superspace V, the superspace of *logarithmic quantum fields*,

$$\mathrm{LFie}(V) = \mathrm{LFie}(V)_{\bar{0}} \oplus \mathrm{LFie}(V)_{\bar{1}},$$

is defined by

$$\mathrm{LFie}(V)_{\bar{\alpha}} = \mathrm{Hom}(V_{\bar{0}}, V_{\alpha}((z))[\zeta]) \oplus \mathrm{Hom}(V_{\bar{1}}, V_{\alpha+\bar{1}}((z))[\zeta]), \quad \alpha \in \mathbb{Z}_2. \tag{2.30}$$

Here, $V((z)) = V[[z]][z^{-1}]$ is the space of formal Laurent series in z.

The final ingredient needed to define a logarithmic vertex algebra is the notion of a braiding map on a superspace.

Definition 7 (Braiding Map)**.** Let V be a superspace. Then a *braiding map* $\mathscr{S}$ on V is an even linear map

$$\mathscr{S} \in \mathrm{End}(V) \otimes \mathrm{End}(V).$$

satisfying two conditions. In order to present these conditions succinctly, note that a braiding map may be written as a finite sum

$$\mathscr{S} = \sum_{i=1}^{L} \phi_i \otimes \psi_i, \quad \phi_i, \psi_i \in \mathrm{End}(V). \tag{2.31}$$

Let I be the identity on V, and define the operators $\mathscr{S}_{12}, \mathscr{S}_{23}, \mathscr{S}_{13}$ on $V \otimes V \otimes V$ by

$$\begin{aligned}
\mathscr{S}_{12} &= \sum_{i=1}^{L} \phi_i \otimes \psi_i \otimes I, \\
\mathscr{S}_{23} &= \sum_{i=1}^{L} I \otimes \phi_i \otimes \psi_i, \\
\mathscr{S}_{13} &= \sum_{i=1}^{L} \phi_i \otimes I \otimes \psi_i.
\end{aligned} \tag{2.32}$$

Then $\mathscr{S}$ must satisfy the following conditions.

1. $\mathscr{S}$ is *symmetric*, which means that $\mathscr{S}$ commutes with the transposition $P: V\otimes V \to V\otimes V$, so that $\mathscr{S}P = P\mathscr{S}$. Here, P is defined by $P(a\otimes b) = (-1)^{p(a)p(b)} b\otimes a$.

2. $[\mathscr{S}_{12}, \mathscr{S}_{23}] = [\mathscr{S}_{13}, \mathscr{S}_{23}] = [\mathscr{S}_{12}, \mathscr{S}_{13}] = 0$.

A braiding map $\mathscr{S}$ is called *locally nilpotent* if for all $a, b \in V$, there exists some positive integer r such that $\mathscr{S}^r(a\otimes b) = 0$. Finally, we may present the definition of a logarithmic vertex algebra.

Definition 8 (Logarithmic Vertex Algebra [BV22])**.** A *logarithmic vertex algebra* (logVA) is a vector superspace V furnished with a distinguished even vector $\mathbb{1}$, an even endomorphism T, an even linear map $Y: V \to \mathrm{LFie}(V)$, and a braiding map $\mathscr{S} \in \mathrm{End}(V)\otimes\mathrm{End}(V)$ such that the following axioms hold:

(**vacuum**) $\quad Y(\mathbb{1}, z) = I, \quad Y(a,z)\mathbb{1} \in V[[z]], \quad Y(a,z)\mathbb{1}|_{z=0} = a, \quad T\mathbb{1} = 0,$

(**translation covariance**) $\quad [T, Y(a,z)] = D_z Y(a,z),$

(**nilpotence**) $\quad \mathscr{S}$ is locally nilpotent on $V\otimes V$,

(**locality**) $\quad$ For every $a, b \in V$, there exists $N \geq 0$ such that for all $c \in V$

$$Y(z_1)(I\otimes Y(z_2)) z_{12}^N e^{\theta_{12}\mathscr{S}_{12}}(a\otimes b\otimes c)$$
$$= (-1)^{p(a)p(b)} Y(z_2)(I\otimes Y(z_1)) z_{12}^N e^{\theta_{21}\mathscr{S}_{12}}(b\otimes a\otimes c).$$

(**hexagon**) $\quad$ The following identity is satisfied on $V\otimes V\otimes V$:

$$\mathscr{S}(Y(z)\otimes I) = (Y(z)\otimes I)(\mathscr{S}_{13} + \mathscr{S}_{23}).$$

Above, the notation $Y(z)$ represents the following linear maps and is introduced to simplify notation in the future:

$$Y(z): V\otimes V \to V((z))[\zeta], \qquad X(z): V\otimes V \to V((z))$$
$$Y(z)(a\otimes b) := Y(a,z)b, \qquad X(z)(a\otimes b) := X(a,z)b = Y(a,z)b\,|_{\zeta=0}\,. \tag{2.33}$$

In the definition of a logVA, the braiding map, it turns out, is a linear sum of tensor products of derivations (in the super sense) of the so-called nth-products $\mu_{(n+\mathscr{S})} := \mu_{(n)}$ ($n \in \mathbb{Z}$) of the algebra of the logVA V. These latter products are bilinear products $\mu_{(n)} : V \times V \to V$ defined in terms of the action of the *modes* of a (also defined in a natural way), but now with an action of the braiding map interposed [BV22]. The modes $a_{(n+\mathscr{S})}$ of a vector $a \in V$ are defined via the state-field correspondence Y by

$$X(a,z) := Y(a,z)\,|_{\zeta=0} = \sum_{n\in\mathbb{Z}} a_{(n+\mathscr{S})} z^{-n-1}, \tag{2.34}$$

and the bilinear $(n+\mathscr{S})$-products $\mu_{(n+\mathscr{S})}$ are subsequently defined by

$$\mu_{(n+\mathscr{S})}(a\otimes b) = a_{(n+\mathscr{S})} b. \tag{2.35}$$

In fact, we may write down an explicit expression for the field $Y(a,z)$ as follows [BV22], where $b \in V$:

$$\begin{aligned} Y(a,z)b &= \sum_{\substack{i\geq 0\\ n\in\mathbb{Z}}} \frac{(-1)^i}{i!} \zeta^i z^{-n-1} \mathscr{S}^i a_{(n+\mathscr{S})} b \\ &= \sum_{n\in\mathbb{Z}} z^{-n-1-\mathscr{S}} a_{(n+\mathscr{S})} b. \end{aligned} \tag{2.36}$$

Here, the notation $\mathscr{S}^i a_{(n+\mathscr{S})} b := \mu_{(n)}\big(\mathscr{S}^i (a\otimes b)\big)$ and $z^{-\mathscr{S}} := e^{-\zeta\mathscr{S}}$. With the notations introduced above, we then have the following relations between $Y(z)$ and $Z(x)$:

$$Y(z) = X(z)e^{-\zeta\mathscr{S}}, \qquad X(z) = Y(z)e^{\zeta\mathscr{S}}. \tag{2.37}$$

It should be noted that the definition of a logVA given in [BV22] comes in two equivalent forms, one in terms of fields in End V, and the other in terms of endomorphisms of V, roughly speaking. The Existence Theorem, Theorem 13 below is written in terms of fields, so it is perhaps convenient to state the other definition as well.

First, we will need the notions of $\hat{\mathscr{S}}$-locality for fields in $\mathrm{LFie}(V)$ and the nth product of two logarithmic fields.

Definition 9. For $\mathcal{V} \subset \mathrm{LFie}(V)$ a subspace and $\hat{\mathscr{S}}$ a locally nilpotent braiding map on $\mathcal{V}$, two logarithmic fields $a, b \in \mathcal{V}$ are called *$\hat{\mathscr{S}}$-local* if

$$\mu\left(z_{12}^{N} e^{\vartheta_{12}\hat{\mathscr{S}}} a(z_1) \otimes b(z_2)\right) = (-1)^{p(a)p(b)} \mu\left(z_{21}^{N} e^{\vartheta_{12}\hat{\mathscr{S}}} b(z_2) \otimes a(z_1)\right), \tag{2.38}$$

where μ is the composition of fields.

Definition 10. If $\hat{\mathscr{S}} = \sum_{i=1}^{L} \Phi_i \otimes \Psi_i$ is a braiding map on $\mathrm{LFie}(V)$, then a subspace $\mathcal{V} \subset \mathrm{LFie}(V)$ is called $\hat{\mathscr{S}}$-local if

(1) $\Phi_i \mathcal{V} \subset \mathcal{V}, \quad \Psi_i \mathcal{V} \subset \mathcal{V}, \quad 1 \leq i \leq L,$

(2) The restriction of $\hat{\mathscr{S}}$ to $\mathcal{V} \otimes \mathcal{V}$ is locally nilpotent,

(3) every two fields $a, b \in \mathcal{V}$ are $\hat{\mathscr{S}}$-local

We also define a product on fields.

Definition 11. Let $a, b \in \mathcal{V}$ be two local logarithmic fields, with N from Definition 9. For each $n \in \mathbb{Z}$, the $(n+\hat{\mathscr{S}})$*-th product* $a_{(n+\hat{\mathscr{S}})}b$ is defined for $v \in V$ by

$$\left(a(z)_{(n+\hat{\mathscr{S}})} b(z)\right) v = D_{z_1}^{(N-1-n)} \mu\left(z_{12}^{N} e^{\vartheta_{12}\hat{\mathscr{S}}} a(z_1, \zeta_1) \otimes b(z_2 \zeta_2)\right) v \Big|_{\substack{z_1 = z_2 = z \\ \zeta_1 = \zeta_2 = \zeta}}. \tag{2.39}$$

This also defines a map

$$\hat{\mu}_{(n)} : \mathcal{V} \otimes \mathcal{V} \to \mathrm{LFie}(V), \qquad \hat{\mu}_{(n)}(a \otimes b) = a_{(n+\hat{\mathscr{S}})} b. \tag{2.40}$$

Definition 12. A logarithmic field $a \in \mathrm{LFie}(V)$ is called *translation covariant* if

$$[T, a(z)] = D_z a(z), \tag{2.41}$$

and denote by $\mathrm{LFie}_T(V)$ the space of translation covariant logarithmic fields.

Definition 13. [BV22] An $\hat{\mathscr{S}}$*-logarithmic vertex algebra* is a vector superspace V (space of states) equipped with an even vector $\mathbb{1} \in V_{\bar{0}}$ (vacuum vector), an even endomorphism $T \in \mathrm{End}(V)_{\bar{0}}$ (translation operator), an even linear map (state-field correspondence)

$$Y_z : V \to \mathrm{LFie}(V), \qquad a \to Y_z(a) = Y(a,z),$$

and a braiding map $\hat{\mathscr{S}}$ on $Y_z(V)$ subject to the following axioms:

$$\begin{aligned}
&(\textbf{vacuum}) && Y(\mathbb{1},z) = I, \quad Y(a,z)\mathbb{1} \in V[\![z]\!], \quad Y(a,z)\mathbb{1}|_{z=0} = a, \quad T\mathbb{1} = 0,\\
&(\textbf{translation covariance}) && [\mathrm{ad}_T \otimes I, \hat{\mathscr{S}}] = [D_z \otimes I, \hat{\mathscr{S}}] = 0, \quad [T, Y(a,z)] = D_z Y(a,z),\\
&(\textbf{locality}) && \text{The space } Y_z(V) \text{ is } \hat{\mathscr{S}}\text{-local},\\
&(\textbf{hexagon}) && \text{For all } n \in \mathbb{Z}, \text{ we have}\\
& && \hat{\mathscr{S}}(\hat{\mu}_{(n)} \otimes I) = (\hat{\mu}_{(n)} \otimes I)(\hat{\mathscr{S}}_{13} + \hat{\mathscr{S}}_{23}) \quad \text{on} \quad Y_z(V)^{\otimes 3}.
\end{aligned}$$

We shall need the notion of a *derivation* of a logVA V, which, simply put, is an endomorphism of V which acts as a derivation (in the algebraic sense) of all of the nth products $\mu_{(n)}$ of V. Explicitly, we have the following definition.

Definition 14 (Derivation of a logVA)**.** Let V be a logVA. An endomorphism $D : V \to V$ is called a *derivation* of V if the following identity holds for all $a, b \in V$ and all $n \in \mathbb{Z}$:

$$D\mu_{(n+\mathscr{S})}(a \otimes b) = \mu_{(n+\mathscr{S})}(Da \otimes b) + (-1)^{p(a)p(D)}\mu_{(n+\mathscr{S})}(a \otimes (Db)) \tag{2.42}$$

The set of derivations of a logVA V will be denoted $\mathrm{Der}(V)$. Let us recall an elementary result from [BV22], which states that for every $D \in \mathrm{Der}(V)$, we have

$$D\mathbb{1} = 0 \qquad \text{and} \qquad [D, T] = 0, \tag{2.43}$$

where T is the translation operator in V.

Properties of logVAs

For ease of reference, it is advantageous to list some of the properties of logVAs, whose statements and proofs are contained in [BV22; BV23]. We list them here without proof and in the form of propositions.

The central identity for our purposes is the Borcherds identity, which follows, and which will be given in two forms—the first in terms of fields, and the second in terms of modes. To express the identity in fields, let us introduce the following generalization of the formal delta function $\delta(z_1,z_2)=\sum_{n\in\mathbb{Z}} z_1^{-n-1}z_2^n$ due to [Bak16]:

$$\delta_{\mathscr{S}}(z_1,z_2):=\delta(z_1,z_2)e^{(\zeta_2-\zeta_1)\mathscr{S}}=\sum_{n\in\mathbb{Z}} z_1^{-n-1-\mathscr{S}}z_2^{n+\mathscr{S}}. \tag{2.44}$$

This generalized delta function satisfies the following property, which will be useful:

Proposition 6. *The following equation holds in a logVA:*

$$Y(z_1)(a\otimes b)\delta(z_1,z_2)=Y(z_2)\delta_{\mathscr{S}}(z_1,z_2)(a\otimes b). \tag{2.45}$$

Proposition 7 (Borcherds Identity, [BV22])**.** *In a logVA V, the following identity holds for every triple $a,b,c\in V$ and every $n\in\mathbb{Z}$:*

$$\begin{aligned}
&\iota_{z_1,z_2}Y(z_1)(I\otimes Y(z_2))z_{12}^n e^{\vartheta_{12}\mathscr{S}_{12}}(a\otimes b\otimes c)\\
&-(-1)^{p(a)p(b)}\iota_{z_2,z_1}Y(z_2)(I\otimes Y(z_1))z_{12}^n e^{\vartheta_{21}\mathscr{S}_{12}}(b\otimes a\otimes c)\\
=&\sum_{j\geq 0}\frac{1}{j!}Y(z_2)(\mu_{(n+j)}\otimes I)D_{z_2}^j\delta_{\mathscr{S}_{13}}(z_1,z_2)(a\otimes b\otimes c).
\end{aligned} \tag{2.46}$$

Equivalently, the $(n+\mathscr{S})$-products satisfy the following identity—the analogue for logVAs of the

Borcherds identity (2.2.1)*—for all* $k, m, n \in \mathbb{Z}$*:*

$$\begin{aligned}
&\sum_{j\geq 0}(-1)^j \mu_{(m+n-j)}\left(I\otimes\mu_{(k+j)}\right)\binom{n+\mathscr{S}_{12}}{j}\\
&-\sum_{j\geq 0}(-1)^{n+j}\mu_{(n+k-j)}\left(I\otimes\mu_{(m+j)}\right)\binom{n+\mathscr{S}_{12}}{j}P\otimes I\\
=&\sum_{j\geq 0}\mu_{(m+k-j)}\left(\mu_{(n+j)}\otimes I\right)\binom{m+\mathscr{S}_{13}}{j}.
\end{aligned}\tag{2.47}$$

The Borcherds identity (2.46) is equivalent to the following identity, which is a version of the *Jacobi identity* in VAs, and which was proved in [BV22].

Proposition 8 (Jacobi identity)**.** *In every logVA* V*, we have the following identity on* $V^{\otimes 3}$*:*

$$\begin{aligned}
&\iota_{z_1,z_2}Y(z_1)(I\otimes Y(z_2))\delta_{\mathscr{S}_{12}}(z_3,z_{12})\\
&-\iota_{z_2,z_1}Y(z_2)(I\otimes Y(z_1))\delta_{\mathscr{S}_{12}}(z_3,z_{12})(P\otimes I)\\
=&\,\iota_{z_1,z_3}Y(z_2)(Y(z_3)\otimes I)\delta_{\mathscr{S}_{13}}(z_{13},z_2).
\end{aligned}\tag{2.48}$$

Taking $n=0$ in (2.47), the following case of the Borcherds identity for a logVA V and $a, b, c \in V$ obtains:

$$\begin{aligned}
&\iota_{z_1,z_2}Y(z_1)(I\otimes Y(z_2))e^{\vartheta_{12}\mathscr{S}_{12}}(a\otimes b\otimes c)\\
&-(-1)^{p(a)p(b)}\iota_{z_2,z_1}Y(z_2)(I\otimes Y(z_1))e^{\vartheta_{21}\mathscr{S}_{12}}(b\otimes a\otimes c)\\
=&\sum_{j\geq 0}\frac{1}{j!}Y(z_2)(\mu_{(j)}\otimes I)D_{z_2}^j\delta_{\mathscr{S}_{13}}(z_1,z_2)(a\otimes b\otimes c)
\end{aligned}\tag{2.49a}$$

or, equivalently in terms of modes:

$$\begin{aligned}
&\sum_{j\geq 0}(-1)^j\mu_{(m-j)}\left(I\otimes\mu_{(k+j)}\right)\binom{\mathscr{S}_{12}}{j}(a\otimes b\otimes c)\\
&-(-1)^{p(a)p(b)}\sum_{j\geq 0}(-1)^j\mu_{(k-j)}\left(I\otimes\mu_{(m+j)}\right)\binom{\mathscr{S}_{12}}{j}(b\otimes a\otimes c)
\end{aligned}\tag{2.49b}$$

$$= \sum_{j\geq 0} \mu_{(m+k-j)} \Big(\mu_{(j)} \otimes I\Big) \binom{m+\mathscr{S}_{13}}{j} (a \otimes b \otimes c).$$

We shall need the specialization of (2.49b) to the case $m, k = 0$, which we write down for future reference below:

$$\begin{aligned} &\sum_{j\geq 0} (-1)^j \mu_{(-j)} \Big(I \otimes \mu_{(j)}\Big) \binom{\mathscr{S}_{12}}{j} (a \otimes b \otimes c) \\ &-(-1)^{p(a)p(b)} \sum_{j\geq 0} (-1)^j \mu_{(-j)} \Big(I \otimes \mu_{(j)}\Big) \binom{\mathscr{S}_{12}}{j} (b \otimes a \otimes c) \\ =&\sum_{j\geq 0} \mu_{(-j)} \Big(\mu_{(j)} \otimes I\Big) \binom{\mathscr{S}_{13}}{j} (a \otimes b \otimes c). \end{aligned} \tag{2.50}$$

If the braiding map $\mathscr{S} : V \otimes V \to V \otimes V$ is given in the form in (2.31), then the action of $\mathscr{S}$ on $a \otimes b$, where $a, b \in V$ is given by

$$\mathscr{S}(a \otimes b) = \sum_{\ell \in [1..L]} (-1)^{p(\psi_\ell)p(a)} \phi_\ell(a) \otimes \psi_\ell(b). \tag{2.51}$$

For reference, we also write down the explicit action of $\mathscr{S}_{12}$ and $\mathscr{S}_{13}$ on $a \otimes b \otimes c \in V \otimes V \otimes V$:

$$\mathscr{S}_{12}(a \otimes b \otimes c) = \sum_{\ell=1}^{L} (-1)^{p(\psi_\ell)p(a)} \big(\phi_\ell(a)\big) \otimes \big(\psi_\ell(b)\big) \otimes c, \tag{2.52}$$

$$\mathscr{S}_{13}(a \otimes b \otimes c) = \sum_{\ell=1}^{L} (-1)^{p(\psi_\ell)\left(p(a)+p(b)\right)} \big(\phi_\ell(a)\big) \otimes b \otimes \big(\psi_\ell(c)\big). \tag{2.53}$$

Proceeding from the above exposition of the various cases and forms of the logVA Borcherds identity (2.47), we now state some useful properties of the $(n+\mathscr{S})$-products $\mu_{(n)} := \mu_{(n+\mathscr{S})}$ of a logVA V.

Proposition 9 ([BV22])**.** *For any logVA V and $a, b \in V$, $n \in \mathbb{Z}$, we have the following relations:*

(1) $\mu_{(-1)}(\mathbb{1} \otimes a) = \mu_{(-1)}(a \otimes \mathbb{1}) = a$,

(2) $\mu_{(n)}(Ta \otimes b) = -n\mu_{(n-1)}(a \otimes b) - \mu_{(n-1)}(\mathscr{S}(a \otimes b))$,

(3) $T\left(\mu_{(n)}(a\otimes b\right) = \mu_{(n)}(Ta\otimes b) + \mu_{(n)}(a\otimes Tb)$,

(4) $\mu_{(n)}(a\otimes b) = -(-1)^{p(a)p(b)}\sum_{j\geq 0}(-1)^{j+n}\frac{1}{j!}T^j\mu_{(n+j)}(b\otimes a)$.

Note that property (3) above implies that the translation operator T is a derivation of the nth products of V. We also give some properties of the component derivations ϕ_i, ψ_i of the braiding map $\mathscr{S}$.

Proposition 10 ([BV22])**.** *For any logVA V with braiding map as in* (2.31) *and any $a, b \in V$, we have:*

(1) $\sum_{i=1}^{L}\phi_i\otimes\psi_i = \sum_{i=1}^{L}(-1)^{p(\phi_i)}\psi_i\otimes\phi_i$,

(2) $[\phi_i,\phi_j] = [\phi_i,\psi_j] = [\psi_i,\psi_j] = 0$,

(3) $\phi_i\mu_{(n)}(a\otimes b) = \mu_{(n)}(\phi_i a\otimes b) + (-1)^{p(\phi_i)p(a)}\mu_{(n)}(a\otimes\phi_i b)$,

(4) $\psi_i\mu_{(n)}(a\otimes b) = \mu_{(n)}(\psi_i a\otimes b) + (-1)^{p(\psi_i)p(a)}\mu_{(n)}(a\otimes\psi_i b)$,

(5) $[T,\phi_i] = [T,\psi_i] = 0$.

Properties 3, 4 in Proposition 10 simply state that ϕ_i, ψ_i are derivations of the nth products in V.

LogVAs also satisfy a skew-symmetry property that will be useful in the sequel. The following properties are proved in [BV22] and are analogous to similar properties in VAs.

Proposition 11. *In any logVA, we have $Y(a,z)\mathbb{1} = e^{zT}a$ and*

$$e^{-z_2T}Y(a,z_1,\zeta_1)e^{z_2T} = \iota_{z_1,z_2}Y(a,z_{12},\zeta_{12}). \tag{2.54}$$

Proposition 12 (Skew-symmetry)**.** *Let V be a logVA with translation operator T, and take $a, b \in V$. Then*

$$Y(a,z,\zeta)b = (-1)^{p(a)p(b)}e^{zT}Y(b,-z,\zeta)a. \tag{2.55}$$

Existence Theorem

An important existence theorem for logVAs upon which the proof of our main result relies is the following one, stated and proved in [BV22].

Theorem 13 (Existence Theorem)**.** Let V be a superspace equipped with an even vector $\mathbb{1} \in V_{\bar{0}}$ and an even linear operator $T \in \mathrm{End}(V)_{\bar{0}}$ such that $T\mathbb{1} = 0$, and let $\mathscr{S}$ be a braiding map on V of the form (2.31) such that

$$[T, \phi_i] = 0 = [T, \psi_i], \qquad 1 \leq i \leq L.$$

Set $\hat{\mathscr{S}} = (\mathrm{ad} \otimes \mathrm{ad})(\mathscr{S})$ and extend it to a braiding map on $\mathrm{LFie}(V)$ by acting only on the coefficients in front of powers of $z^{\pm 1}$ and ζ, in the logarithmic fields. Fix an $\hat{\mathscr{S}}$-local subspace

$$\mathcal{V} = \mathrm{span}\left\{I, a^i(z) \mid i \in J\right\} \subset \mathrm{LFie}_T(V)$$

of translation covariant logarithmic fields, where J is an index set. Suppose that $\mathcal{V}$ is complete in the sense that

$$V = \mathrm{span}\left\{\mathbb{1}, a^{i_1}{}_{(n_1+\mathscr{S})} \cdots a^{i_k}{}_{(n_k+\mathscr{S})}\mathbb{1} \;\mid\; k \geq 1, i_1, \ldots, i_k \in J, n_1, \ldots, n_k \in \mathbb{Z}\right\}.$$

Then $\bar{\mathcal{V}} \subset \mathrm{LFie}_T(V)$ is an $\hat{\mathscr{S}}$-local space containing $\mathcal{V}$, closed under all $(n+\mathscr{S})$-products, and for every $v \in V$, there exists a unique logarithmic field $Y(v,z) \in \bar{\mathcal{V}}$ such that $Y(v,z)\mathbb{1}\,|_{z=0} = v$. Furthermore, $\bar{\mathcal{V}}$ is the maximal $\hat{\mathscr{S}}$-local subspace of $\mathrm{LFie}_T(V)$ containing $\mathcal{V}$.

2.2.4 Non-local Lie Conformal Algebras and Non-local Poisson Vertex Algebras

In this section, we review the definition of a *non-local Poisson vertex algebra* (non-local PVA) given in [DSK13], which extends the notion of a Poisson vertex algebra (PVA). A non-local PVA is built upon the notion of a *non-local Lie conformal algebra* (non-local LCA) with an additional structure. Based on the connection established by Bakalov and Villarreal in [BV23; BV24] between logVAs and non-local PVAs, we will be able to posit the existence of new examples of non-local PVAs at the end of our results in Chapter 3.

Before we present the definitions of non-local LCAs and non-local PVAs, let us briefly review the older notions of LCA and PVA.

Definition 15. A *Lie conformal algebra (LCA)* (see e.g. [BDSK21]) is an LCA V, equipped with a linear map $V \otimes V \to \mathbb{C}[\lambda] \otimes V$ called a λ*-bracket* and denoted by $a \otimes b \mapsto \{a_\lambda b\}$ satisfying the following properties for any $a, b, c \in V$:

$$\begin{aligned}
&(\textit{sesquilinearity}) && \{\partial a_\lambda b\} = -\lambda \{a_\lambda b\}, \qquad \{a_\lambda \partial b\} = (\lambda + \partial)\{a_\lambda b\}, \\
&(\textit{skew symmetry}) && \{b_\lambda a\} = -\{a_{-\lambda-\partial} b\}, \\
&(\textit{Jacobi identity}) && \{a_\lambda \{b_\mu c\}\} = \{\{a_\lambda b\}_{\lambda+\mu} c\} + \{b_\mu \{a_\lambda c\}\}.
\end{aligned}$$

Definition 16. A *Poisson vertex algebra (PVA)* (see e.g. [BDSK21]) is an LCA V with an additional commutative associative multiplication given by juxtaposition ($ab \in V$ for $a, b \in V$). V has a unit $1 \in V$ with respect to this multiplication, and ∂ is a derivation of it for which the Leibniz rule holds. That is, we have

$$\begin{aligned}
&(\textit{derivation}) && \partial(ab) = (\partial a)b + a(\partial b), \\
&(\textit{Leibniz rule}) && \{a_\lambda bc\} = \{a_\lambda b\}\, c + b\, \{a_\lambda c\}.
\end{aligned}$$

The difference between these older notions and their non-local counterparts is that the λ-bracket takes values in $V((\lambda^{-1}))$ rather than $V[\lambda]$. Explicitly, we have the definitions:

Definition 17. A *non-local Lie conformal algebra* (non-local LCA) [BV23] is a $\mathbb{C}[\partial]$-module V

equipped with an admissible λ-bracket

$$\{\cdot_\lambda\cdot\} : V \times V \to V(\!(\lambda^{-1})\!) \tag{2.56}$$

which satisfies

$$\begin{aligned}
&\big(\textit{sesquilinearity}\big) \quad \{\partial a_\lambda b\} = -\lambda \{a_\lambda b\}, \qquad \{a_\lambda \partial b\} = (\lambda + \partial)\{a_\lambda b\},\\
&\big(\textit{skew symmetry}\big) \quad \{b_\lambda a\} = -\{a_{-\lambda-\partial}\, b\},\\
&\big(\textit{Jacobi identity}\big) \quad \big\{a_\lambda \big\{b_\mu c\big\}\big\} = \big\{\{a_\lambda b\}_{\lambda+\mu} c\big\} + \big\{b_\mu \{a_\lambda c\}\big\},
\end{aligned}$$

where the admissibility of the λ-bracket is a technical condition guaranteeing the well-definition of the compositions in the Jacobi identity above.

Definition 18. A *non-local Poission vertex algebra* (non-local PVA) [DSK13] is a non-local LCA V with an additional commutative associative multiplication given by juxtaposition ($ab \in V$ for $a, b \in V$). V has a unit $1 \in V$ with respect to this multiplication, and ∂ is a derivation of it for which the Leibniz rule holds. That is, we have

$$\begin{aligned}
&(\textit{derivation}) \quad \partial(ab) = (\partial a)b + a(\partial b),\\
&(\textit{Leibniz rule}) \quad \{a_\lambda bc\} = \{a_\lambda b\}\, c + b\, \{a_\lambda c\}.
\end{aligned}$$

Let V now be a *filtered logVA*; that is, we have a sequence of subspaces

$$\{0\} = \cdots = F^{-1}V \subset F^1 V \subset F^2 V \subset \cdots, \quad \bigcup_{n\geq 0} F^n V = V \tag{2.57}$$

such that the following properties hold for $m, n \geq 0$ and $\mathscr{S}$ the braiding map on V:

(1) $\mathbb{1} \in F^0 V$,

(2) $T(F^n V) \subset F^n V$,

(3) $\mathscr{S}(F^m V \otimes F^n V) \subset F^{m+n-1}(V \otimes V) := \sum_{k=0}^{m+n-1} F^k V \otimes F^{m+n-1-k} V,$

(4) $\mu_{(j)}(F^m V \otimes F^n V) \subset \begin{cases} F^{m+n} V, & j < 0, \\ F^{m+n-1} V, & j \geq 0. \end{cases}$

From the filtered logVA V, form the *associated graded algebra* $\operatorname{gr} V := \bigoplus_{n\in\mathbb{Z}} \operatorname{gr}^n V$, where $\operatorname{gr}^n V = F^n V / F^{n-1} V$. Then a theorem proved in [BV23] tells us that $\operatorname{gr} V$ is a non-local PVA, where $1 \in \operatorname{gr} V^0 = F^0 V$ is $\mathbb{1}$, ∂ is descended from T in V, the commutative associative product on $\operatorname{gr} V$ is descended from $\mu_{(-1)}$, and the λ-bracket on $\operatorname{gr} V$ is given by:

$$\{a_\lambda b\} := \sum_{n\geq 0} \frac{\lambda^n}{n!} \mu_{(n)}(a \otimes b) + \mu_{(-1)}\left(\mathscr{S}\left(\frac{1}{\lambda+\partial} a \otimes b\right)\right). \tag{2.58}$$

CHAPTER

3

MAIN RESULTS

The main result of the present work implies the existence of logVAs under certain conditions on the nonnegative nth products, and allows for the construction of classes of examples of logVAs heretofore unknown. Additionally, a version of the PBW Theorem is obtained, as well as an existence theorem for non-local PVAs under assumptions on a generating subspace coinciding with those from our main result, Theorem 17. Specializing the existence theorem for non-local PVAs to the logVAs whose existence and non-triviality are proved in our main results, we also obtain non-local PVA structures from each instance of a new logVA so constructed.

3.1 Preliminary Observations

Let us first make some observations regarding the symmetries of the Borcherds identity. These will imply that the relations in our constructed logVA will only need to be imposed on ordered triples of *distinct* generators. We have the following result, whose proof mimics that of a similar result for VAs found in [FHL93].

Proposition 14. *Let V be a superspace equipped with a state-field correspondence $Y : V \to V((z))[\zeta]$, a locally nilpotent braiding map $\mathscr{S}$, an even endomorphism T, and satisfying the skew-symmetry property* (2.55)*. If the Borcherds identity* (2.46) *with $n = 0$ is satisfied in V for $a \otimes b \otimes c$, then it is also satisfied for $\sigma(a) \otimes \sigma(b) \otimes \sigma(c)$, where σ is any permutation of the triple (a, b, c).*

Proof. Recall the Jacobi identity (2.48), which is equivalent to the Borcherds identity when applied to $a \otimes b \otimes c \in V^{\otimes 3}$; at $n = 0$:

$$\begin{aligned}
&\iota_{z_1,z_2} Y(z_1)(I \otimes Y(z_2))\delta_{\mathscr{S}_{12}}(z_3, z_{12})(a \otimes b \otimes c) \\
&-\iota_{z_2,z_1} Y(z_2)(I \otimes Y(z_1))\delta_{\mathscr{S}_{12}}(z_3, z_{12})(P \otimes I)(a \otimes b \otimes c) \\
=&\ \iota_{z_1,z_3} Y(z_2)(Y(z_3) \otimes I)\delta_{\mathscr{S}_{13}}(z_{13}, z_2)(a \otimes b \otimes c).
\end{aligned} \tag{3.1}$$

Let $\sigma : (a, b, c) \to (b, a, c)$, and observe that we may rewrite the skew-symmetry property in the following "operator" form in terms of the state-field correspondence Y (recall that $Y(z)(a \otimes b) = Y(a, z)b$ and $P(a \otimes b) = (-1)^{p(a)p(b)} b \otimes a$):

$$Y(z) = e^{zT} Y(-z)P. \tag{3.2}$$

Applying skew-symmetry to the right-hand term in the Jacobi identity, we obtain

$$Y(z_2)(Y(z_3) \otimes I)\delta_{\mathscr{S}_{13}}(z_{13}, z_2)(a \otimes b \otimes c) = (-1)^{p(a)p(b)} Y(z_2)\left(e^{z_3 T} Y(-z_3) \otimes I\right)(b \otimes a \otimes c). \tag{3.3}$$

To the right-hand side of the latter, applying the property of the delta function from (2.45) and properties of the usual delta function and rearranging, we obtain the Jacobi identity for $b \otimes a \otimes c$.

A similar approach will yield the identity for $a \otimes c \otimes b$. It follows that the Jacobi identity for $a \otimes b \otimes c$ implies it for any permutation of a, b, c. □

We also characterize (2.46) at $n = 0$ in relation to the translation operator T in a putative logVA. Specifically, we have the following result.

Proposition 15. *Let V be a vector superspace with an even endomorphism T, a translation covariant state-field correspondence $Y : V \to V(\!(z)\!)[\zeta]$, and a locally nilpotent braiding map $\mathscr{S}$. Suppose that the Borcherds identity* (2.46) *at $n = 0$ is satisfied in V. Then it is invariant under ad T.*

Proof. We may apply ad T directly to (2.49a) and use the relation (3.29b). We work term-by-term to reduce clutter. First, applying ad $T := [T, \cdot]$ to the first term on the left-hand side of (2.49a), we have

$$\begin{aligned}
&\operatorname{ad} T\left(\iota_{z_1,z_2} Y(z_1)(I \otimes Y(z_2))e^{\vartheta_{12}\mathscr{S}_{12}}\right)(a \otimes b \otimes c) \\
&= \iota_{z_1,z_2}\left([T, Y(z_1)](I \otimes Y(z_2))e^{\vartheta_{12}\mathscr{S}_{12}} + Y(z_1)(I \otimes [T, Y(z_2)])e^{\vartheta_{12}\mathscr{S}_{12}}\right)(a \otimes b \otimes c) \\
&= \iota_{z_1,z_2}\left(D_{z_1} Y(z_1)(I \otimes Y(z_2))e^{\vartheta_{12}\mathscr{S}_{12}} + Y(z_1)(I \otimes D_{z_2} Y(z_2))e^{\vartheta_{12}\mathscr{S}_{12}}\right)(a \otimes b \otimes c) \\
&= (D_{z_1} + D_{z_2})\left(\iota_{z_1,z_2}(Y(z_1) \otimes Y(z_2))e^{\vartheta_{12}\mathscr{S}_{12}}\right)(a \otimes b \otimes c),
\end{aligned}$$

and by a similar computation, we have for the second left-hand term

$$\begin{aligned}
&\operatorname{ad} T\left(-(-1)^{p(a)p(b)}\iota_{z_2,z_1} Y(z_2)(I \otimes Y(z_1))e^{\vartheta_{21}\mathscr{S}_{12}}\right)(b \otimes a \otimes c) \\
&= -(-1)^{p(a)p(b)}\iota_{z_2,z_1}\left([T, Y(z_2)](I \otimes Y(z_1))e^{\vartheta_{21}\mathscr{S}_{12}} + Y(z_2)(I \otimes [T, Y(z_1)])e^{\vartheta_{21}\mathscr{S}_{12}}\right)(b \otimes a \otimes c) \\
&= -(-1)^{p(a)p(b)}\iota_{z_2,z_1}\left(D_{z_2} Y(z_2)(I \otimes Y(z_1))e^{\vartheta_{21}\mathscr{S}_{12}} + Y(z_2)(I \otimes D_{z_1} Y(z_1))e^{\vartheta_{21}\mathscr{S}_{12}}\right)(b \otimes a \otimes c) \\
&= -(-1)^{p(a)p(b)}\left((D_{z_2} + D_{z_1})\iota_{z_2,z_1}(Y(z_2) \otimes Y(z_1))e^{\vartheta_{21}\mathscr{S}_{12}}\right)(b \otimes a \otimes c).
\end{aligned}$$

For the right-hand term, we have first, upon integrating by parts,

$$
\begin{aligned}
&\sum_{j\geq 0} Y(z_2)(\mu_{(j)}\otimes I)D_{z_2}^{(j)}\delta_{\mathscr{S}_{13}}(z_1,z_2)(a\otimes b\otimes c)\\
&=\sum_{j\geq 0} Y(z_2)(\mu_{(j)}\otimes I)(-1)^j D_{z_1}^{(j)}\delta_{\mathscr{S}_{13}}(z_1,z_2)(a\otimes b\otimes c)\\
&=\mathrm{Res}_{z_3}\left(Y(z_2)(X(z_3)\otimes I)e^{-z_3D_{z_1}}\delta_{\mathscr{S}_{13}}(z_1,z_2)(a\otimes b\otimes c)\right)\\
&=\mathrm{Res}_{z_3}\iota_{z_1,z_3}\left(Y(z_2)(Y(z_3)\otimes I)z_3^{\mathscr{S}_{12}}\delta_{\mathscr{S}_{13}}(z_{13},z_2)(a\otimes b\otimes c)\right),
\end{aligned}
$$

where we used the divided-powers notation $D_z^{(j)} := D_z^j/j!$. In the second step above, we have used the expansion of $X(z_3)$ in terms of the product $\mu_{(n)}$, in other words, the mode expansion. In the last step, we recover $Y(z_3)$ from $X(z_3)$ in the standard way. At this point, we apply ad T to obtain

$$
\begin{aligned}
&\mathrm{ad}\ T\left(\sum_{j\geq 0} Y(z_2)(\mu_{(j)}\otimes I)D_{z_2}^{(j)}\delta_{\mathscr{S}_{13}}(z_1,z_2)(a\otimes b\otimes c)\right)\\
&=\mathrm{Res}_{z_3}\iota_{z_1,z_3}([T,Y(z_2)](Y(z_3)\otimes I)+Y(z_2)([T,Y(z_3)]\otimes I))\,z_3^{\mathscr{S}_{12}}\delta_{\mathscr{S}_{13}}(z_{13},z_2)(a\otimes b\otimes c)\\
&=\mathrm{Res}_{z_3}\iota_{z_1,z_3}\left(D_{z_2}Y(z_2)(Y(z_3)\otimes I)+Y(z_2)(D_{z_3}Y(z_3)\otimes I)\right)z_3^{\mathscr{S}_{12}}\delta_{\mathscr{S}_{13}}(z_{13},z_2)(a\otimes b\otimes c)\\
&=\mathrm{Res}_{z_3}\iota_{z_1,z_3}\left((D_{z_2}+D_{z_3})Y(z_2)(Y(z_3)\otimes I)+Y(z_2)(Y(z_3)\otimes I)\right)z_3^{\mathscr{S}_{12}}\delta_{\mathscr{S}_{13}}(z_{13},z_2)(a\otimes b\otimes c).
\end{aligned}
$$

Combining the terms together, the claim is now apparent. □

3.2 Forward Direction of Existence Theorem

We begin by first describing an identity satisfied in a logVA under certain conditions on its nonnegative nth products. Subsequently, we use the given condition and the Borcherds identity to contruct a logVA satisfying these conditions by applying the Existence Theorem (Theorem 13).

Proposition 16. *Let V be a logVA generated under nth products by the subspace U, and let*

$\mathscr{S} = \sum_{i=1}^{L} \phi_i \otimes \psi_i$ be the braiding map for V, with $\phi_i, \psi_i \in Der\, V$ and L a positive integer. Suppose $\phi_i(U), \psi_i(U) \subset U$. Suppose further that $\mu_{(0+\mathscr{S})}(U \otimes U) \in U$, $\mu_{(1+\mathscr{S})}(U \otimes U) \in \mathbb{C}\mathbb{1}$, and $\mu_{(n+\mathscr{S})}(U \otimes U) = 0$ for $n \geq 2$. Then $\mu_{(0)} := \mu_{(0+\mathscr{S})}$ defines an even skew super symmetric bilinear product $[\![\cdot,\cdot]\!] : U \times U \to U$ given by $[\![a, b]\!] = \mu_{(0)}(a \otimes b)$, $\mu_{(1)} := \mu_{(1+\mathscr{S})}$ defines an even super symmetric bilinear product $(\cdot|\cdot) : U \times U \to \mathbb{C}$ defined by $(a \,|\, b)\mathbb{1} = \mu_{(1)}(a \otimes b)$ which is invariant under $[\![\cdot,\cdot]\!]$, and the following identity holds for all $a, b, c \in U$:

$$[\![a, [\![b, c]\!]]\!] - (-1)^{p(a)p(b)} [\![b, [\![a, c]\!]]\!] - [\![[\![a, b]\!], c]\!] = \tag{3.4}$$
$$\sum_{\ell=1}^{L} (-1)^{p(\psi_\ell)\left(p(a)+p(b)\right)} \Big((\phi_\ell a \Big| b)\psi_\ell c + (-1)^{p(\psi_\ell)p(b)} \big(\psi_\ell b \Big| c)\phi_\ell a - (-1)^{p(a)p(b)+p(\psi_\ell)p(a)} \big(\psi_\ell a \Big| c)\phi_\ell b \Big).$$

Proof. The fact (see (2.43)) that $T\mathbb{1} = 0$, combined with the assumption that $\mu_{(n)} = 0$ for $n \geq 2$, when taken in conjunction with Property 4 of the same proposition for $n = 0$ and $n = 1$, establish the super skew symmetry and super symmetry of $[\![\cdot,\cdot]\!]$ and $(\cdot|\cdot)$, respectively. These two products are already bilinear by assumption. We show the computation proving the super skew symmetry of $\mu_{(0)}$; a similar computation establishes the super symmetry of $\mu_{(1)}$. Let $n = 0$ in Property 4 of Proposition 9. Then we have

$$\begin{aligned}
\mu_{(0)}(a \otimes b) &= -(-1)^{p(a)p(b)} \sum_{j \geq 0} (-1)^j \frac{1}{j!} T^j \mu_{(j)}(b \otimes a) \\
&= -(-1)^{p(a)p(b)} \sum_{j=0}^{1} (-1)^j \frac{1}{j!} T^j \mu_{(j)}(b \otimes a) && \big(\mu_{(n)} = 0 \text{ for } n \geq 2\big) \\
&= -(-1)^{p(a)p(b)} \big(\mu_{(0)}(b \otimes a) - T\mu_{(1)}(b \otimes a)\big) \\
&= -(-1)^{p(a)p(b)} \big(\mu_{(0)}(b \otimes a) - T(b \,|\, a)\mathbb{1}\big). && \big(\mu_{(1)}(U \otimes U) \subset \mathbb{C}\mathbb{1}\big) \\
&= -(-1)^{p(a)p(b)} \mu_{(0)}(b \otimes a) && (T\mathbb{1} = 0).
\end{aligned}$$

From now on, we will substitute the symbols $[\![a, b]\!]$ and $(a \,|\, b)\mathbb{1}$ for the super skew symmetric and symmetric products $\mu_{(0)}(a \otimes b)$ and $\mu_{(1)}(a \times b)$, respectively, wherever convenient.

Consider the case $m, k = 0$ in the Borcherds $n = 0$ identity as in (2.50), which we repeat below for ease of reference:

$$
\begin{aligned}
&\sum_{j\geq 0}(-1)^j\mu_{(-j)}\Big(I\otimes\mu_{(j)}\Big)\binom{\mathscr{S}_{12}}{j}(a\otimes b\otimes c)\\
&-(-1)^{p(a)p(b)}\sum_{j\geq 0}(-1)^j\mu_{(-j)}\Big(I\otimes\mu_{(j)}\Big)\binom{\mathscr{S}_{12}}{j}(b\otimes a\otimes c)\\
=&\sum_{j\geq 0}\mu_{(-j)}\Big(\mu_{(j)}\otimes I\Big)\binom{\mathscr{S}_{13}}{j}(a\otimes b\otimes c).
\end{aligned}
\tag{3.5}
$$

Assume now that $a, b, c \in U$ in (2.50). Note that by the assumption that the derivations ϕ_i, ψ_i composing $\mathscr{S}$ be U-invariant, and the form of the expansions of the powers of $\mathscr{S}_{12}$ and $\mathscr{S}_{13}$ (see (2.52) and (2.53)), it follows that the inner sums on both sides of (3.5) are members of $U\otimes U\otimes U$ whenever $a, b, c \in U$. Consequently, applying $I\otimes\mu_{(j)}$ and $\mu_{(j)}\otimes I$ to the summands in the inner sums of the left- and right-hand sides of (3.5), respectively, will yield zero for $j\geq 2$ by the assumptions on $\mu_{(n)}$. Hence, in this setting, the terms of (3.5) take simpler forms. We simplify each sum of sums individually before putting them together. We have

$$
\begin{aligned}
&\sum_{j\geq 0}(-1)^j\mu_{(-j)}\Big(I\otimes\mu_{(j)}\Big)\binom{\mathscr{S}_{12}}{j}(a\otimes b\otimes c)\\
=\;&\mu_{(0)}\big(I\otimes\mu_{(0)}\big)(a\otimes b\otimes c)-\mu_{(-1)}\big(I\otimes\mu_{(1)}\big)\mathscr{S}_{12}(a\otimes b\otimes c)\\
=\;&[\![a,[\![b,c]\!]]\!]-\mu_{(-1)}\big(I\otimes\mu_{(1)}\big)\sum_{\ell=1}^{L}(-1)^{p(\psi_\ell)p(a)}\big(\phi_\ell a\big)\otimes\big(\psi_\ell b\big)\otimes c\\
=\;&[\![a,[\![b,c]\!]]\!]-\sum_{\ell=1}^{L}(-1)^{p(\psi_\ell)p(a)}\big(\psi_\ell b\,\big|\,c\big)\mu_{(-1)}\big(\big(\phi_\ell a\big)\otimes\mathbb{1}\big)\\
=\;&[\![a,[\![b,c]\!]]\!]-\sum_{\ell=1}^{L}(-1)^{p(\psi_\ell)p(a)}\big(\psi_\ell b\,\big|\,c\big)\big(\phi_\ell a\big)
\end{aligned}
\tag{3.6}
$$

for the first double sum on the left-hand side of (3.5). In the above simplifications, we have used the expansion (2.52) of the action of $\mathscr{S}_{12}$, the assumed properties of $\mu_{(0)}$ and $\mu_{(1)}$ in the current proposition, and Property 1 of Proposition 9. By a similar series of steps, we obtain for the second double sum on the left-hand side of (3.5)

$$
\begin{aligned}
&-(-1)^{p(a)p(b)}\sum_{j\geq 0}(-1)^j\mu_{(-j)}\Big(I\otimes\mu_{(j)}\Big)\binom{\mathscr{S}_{12}}{j}(b\otimes a\otimes c)\\
&=\ \mu_{(0)}\big(I\otimes\mu_{(0)}\big)(b\otimes a\otimes c)-\mu_{(-1)}\big(I\otimes\mu_{(1)}\big)\mathscr{S}_{12}(b\otimes a\otimes c)\\
&=\cdots=\ [\![b,[\![a,c]\!]]\!]-\sum_{\ell=1}^{L}(-1)^{p(\psi_\ell)p(b)}\big(\psi_\ell a\,\big|\,c\big)\big(\phi_\ell b\big).
\end{aligned}
\tag{3.7}
$$

Finally, the right-hand side of (3.5) may be simplified in a very similar manner to obtain

$$
\begin{aligned}
&\sum_{j\geq 0}\mu_{(-j)}\Big(\mu_{(j)}\otimes I\Big)\binom{\mathscr{S}_{13}}{j}(a\otimes b\otimes c)\\
&=\ \mu_{(0)}\big(\mu_{(0)}\otimes I\big)(a\otimes b\otimes c)+\mu_{(-1)}\big(\mu_{(1)}\otimes I\big)\mathscr{S}_{13}(a\otimes b\otimes c)\\
&=\ [\![[\![a,b]\!],c]\!]+\mu_{(-1)}\big(\mu_{(1)}\otimes I\big)\sum_{\ell=1}^{L}(-1)^{p(\psi_\ell)\left(p(a)+p(b)\right)}\big(\phi_\ell a\big)\otimes b\otimes\big(\psi_\ell c\big)\\
&=\ [\![[\![a,b]\!],c]\!]+\sum_{\ell=1}^{L}(-1)^{p(\psi_\ell)\left(p(a)+p(b)\right)}\big(\phi_\ell a\,\big|\,b\big)\mu_{(-1)}\big(\mathbb{1}\otimes\big(\psi_\ell c\big)\big)\\
&=\ [\![[\![a,b]\!],c]\!]+\sum_{\ell=1}^{L}(-1)^{p(\psi_\ell)\left(p(a)+p(b)\right)}\big(\phi_\ell a\,\big|\,b\big)\big(\psi_\ell c\big).
\end{aligned}
\tag{3.8}
$$

Replacing the expressions in (3.5) by their final equivalent expressions in (3.6), (3.7) and (3.8) and rearranging, we obtain (3.4) as claimed.

Finally, we prove the invariance of the bilinear form with respect to the double bracket. Recall that for any derivation D of a logVA V, we have $D\mathbb{1}=0$. This implies that $\mathscr{S}(a\otimes\mathbb{1})=\mathscr{S}(\mathbb{1}\otimes b)=0$ for any $a,b\in V$, and by Proposition 9 (2) with $a=\mathbb{1}$ and b arbitrary, and $n=1$ and $n=2$, respectively we find that $[\![a,\mathbb{1}]\!]=0$ and $(a\,|\,\mathbb{1})=0$ for any $a\in V$. The Borcherds identity at $n=0$ for $m=0$ and $k=1$ is the identity

$$
\begin{aligned}
&\sum_{j\geq 0}(-1)^j\mu_{(-j)}\Big(I\otimes\mu_{(1+j)}\Big)\binom{\mathscr{S}_{12}}{j}(a\otimes b\otimes c)\\
&-(-1)^{p(a)p(b)}\sum_{j\geq 0}(-1)^j\mu_{(1-j)}\Big(I\otimes\mu_{(j)}\Big)\binom{\mathscr{S}_{12}}{j}(b\otimes a\otimes c)
\end{aligned}
\tag{3.9}
$$

$$= \sum_{j\geq 0} \mu_{(1-j)} \Big(\mu_{(j)} \otimes I\Big) \binom{\mathscr{S}_{13}}{j} (a \otimes b \otimes c).$$

By the assumptions on the nonnegative nth products and the above discussion, the only terms that do not vanish are the terms corresponding to $j = 0$ in the second sum on the left-hand side, and the term corresponding to $j = 0$ on the right-hand side, giving

$$-(-1)^{p(a)p(b)} \big(b \,\big|\, [\![a, c]\!] \big) = \big([\![a, b]\!] \,\big|\, c \big), \tag{3.10}$$

which immediately gives the claimed invariance after applying the skew symmetry of the double bracket. □

In order to prove a sort of converse to Proposition 16, we will need to impose conditions on the nth products of the generators when acting on elements $a \otimes b \otimes v$, where $a, b \in U$ but $v \in V$ is an arbitrary element of the putative logVA V. To this end, we require the Borcherds identity for $n = 0$, (2.49b), and we expand this identity in terms of the derivations ϕ_i, ψ_i, which compose the braiding map $\mathscr{S}$. The resulting expressions are somewhat complicated, so it is expedient to introduce some space-saving notation.

Defining for convenience the combinatorial functions

$$h_{m,j}(k) = \sum_{\substack{S \subseteq [m-j+1, m] \\ |S| = k}} \prod_{\sigma \in S} \sigma \tag{3.11}$$

$$\tilde{h}(j) = \sum_{i=1}^{j} \frac{1}{i} \qquad \text{for } j \geq 1, \tag{3.12}$$

where sums over empty sets are always interpreted as zero and products over empty sets are interpreted as one, and an interval $[a, b]$ with $a > b$ is interpreted as the empty set, we may rewrite (2.49b) as

$$\sum_{j\geq 0} (-1)^j \mu_{(m-j)} \Big(I \otimes \mu_{(k+j)}\Big) \frac{1}{j!} \sum_{i=0}^{j} h_{0,j}(i) \mathscr{S}_{12}^{j-i} (a \otimes b \otimes c)$$

$$-(-1)^{p(a)p(b)}\sum_{j\geq 0}(-1)^j\mu_{(k-j)}\Big(I\otimes\mu_{(m+j)}\Big)\frac{1}{j!}\sum_{i=0}^{j}h_{0,j}(i)\mathscr{S}_{12}^{j-i}(b\otimes a\otimes c) \tag{3.13}$$
$$=\sum_{j\geq 0}\mu_{(m+k-j)}\Big(\mu_{(j)}\otimes I\Big)\frac{1}{j!}\sum_{i=0}^{j}h_{m,j}(i)\mathscr{S}_{13}^{j-i}(a\otimes b\otimes c).$$

Additionally, by defining the shorthand notations

$$\ell\in[1,L]^n,\ n\geq 1 \quad\Leftrightarrow\quad \ell=(\ell_1,\dots,\ell_n)\quad\text{and}\quad \ell_i\in[1,L], \tag{3.14}$$
$$\phi_{\ell_0}:=\mathrm{Id}, \tag{3.15}$$
$$\phi_{\ell_i}^{\ell_j}:=\phi_{\ell_j}\cdots\phi_{\ell_i}, \tag{3.16}$$

we have explicitly the following expressions for the actions of the powers of $\mathscr{S}_{12}$ and $\mathscr{S}_{13}$ on an arbitrary tensor $a\otimes b\otimes c\in V\otimes V\otimes V$:

$$\mathscr{S}_{12}^{n}(a\otimes b\otimes c)=\sum_{\ell\in[1,L]^n}(-1)^{\sum_{i=1}^{n}p(\psi_{\ell_i})p\left(\phi_{\ell_0}^{\ell_{i-1}}(a)\right)}\Big(\phi_{\ell_1}^{\ell_n}(a)\Big)\otimes\Big(\psi_{\ell_1}^{\ell_n}(b)\Big)\otimes c, \tag{3.17}$$
$$\mathscr{S}_{13}^{n}(a\otimes b\otimes c)=\sum_{\ell\in[1,L]^n}(-1)^{\sum_{i=1}^{n}p(\psi_{\ell_i})\left(p\left(\phi_{\ell_0}^{\ell_{i-1}}(a)\right)+p(b)\right)}\Big(\phi_{\ell_1}^{\ell_n}(a)\Big)\otimes b\otimes\Big(\psi_{\ell_1}^{\ell_n}(c)\Big). \tag{3.18}$$

Equations (3.17) and (3.18) allow us to give the following useful and explicit form of (3.13), where $a,b\in V$, $a_n:=a_{(n+\mathscr{S})}$, $\binom{m}{j}:=\frac{m(m-1)\cdots(m-j+1)}{j!}$ for $j\geq 1$ and $\binom{m}{0}:=1-\delta_{m,0}$, and the equation is rearranged to express the supercommutator of the modes, $[a_m,b_k]=a_mb_k-(-1)^{p(a)p(b)}b_ka_m$, in terms of other modes:

$$[a_m,b_k]=\sum_{j\geq 0}\binom{m}{j}\big(a_jb\big)_{m+k-j}$$
$$+\sum_{j\geq 1}\sum_{n=1}^{j}\sum_{\ell\in[1,L]^n}\tfrac{h_{m,j}(j-n)}{j!}(-1)^{\sum_{i=1}^{n}p(\psi_{\ell_i})\left(p\left(\phi_{\ell_0}^{\ell_{i-1}}a\right)+p(b)\right)}\Big(\Big(\phi_{\ell_1}^{\ell_n}a\Big)_j b\Big)_{m+k-j}\psi_{\ell_1}^{\ell_n}$$
$$-\sum_{j\geq 1}\sum_{n=1}^{j}\sum_{\ell\in[1,L]^n}\tfrac{(-1)^jh_{0,j}(j-n)}{j!}(-1)^{\sum_{i=1}^{n}p(\psi_{\ell_i})p\left(\phi_{\ell_0}^{\ell_{i-1}}a\right)}\Big(\phi_{\ell_1}^{\ell_n}a\Big)_{m-j}\Big(\psi_{\ell_1}^{\ell_n}b\Big)_{k+j} \tag{3.19}$$
$$+(-1)^{p(a)p(b)}\sum_{j\geq 1}\sum_{n=1}^{j}\sum_{\ell\in[1,L]^n}\tfrac{(-1)^jh_{0,j}(j-n)}{j!}(-1)^{\sum_{i=1}^{n}p(\psi_{\ell_i})p\left(\phi_{\ell_0}^{\ell_{i-1}}b\right)}\Big(\phi_{\ell_1}^{\ell_n}b\Big)_{k-j}\Big(\psi_{\ell_1}^{\ell_n}a\Big)_{m+j}.$$

3.3 Existence Theorem

The "converse" of Proposition 16, which is the central theorem (Theorem 17) this work, constructs a logVA V by imposing relations derived from (3.19), along with the necessary identities for the braiding map, the two bilinear products, the endomorphisms ϕ_i and ψ_i, and their status as derivations of all the nth products on a suitably chosen tensor algebra to obtain an associative algebra $\mathcal{A}$ whose module will serve as V.

Once the algebra $\mathcal{A}$ and its module V have been defined, we verify that they satisfy all the necessary properties required in order to apply the Existence Theorem (Theorem 13) and obtain a logVA. We break this process into smaller steps in the proof of Theorem 17 in order to facilitate the exposition. The proof is modeled on a construction of an algebra subject to relations obtained in [BV22] from an LCFT due to Gurarie and Ludwig [Gur13; GL05].

Theorem 17. *Let U be a superspace endowed with an even skew super symmetric bilinear product $[\![\cdot,\cdot]\!] : U \times U \to U$ and an even super symmetric bilinear product $(\cdot|\cdot) : U \times U \to \mathbb{C}$. Let $L \geq 1$ be a positive integer, and let ϕ_i, ψ_i be nilpotent endomorphisms of U for $1 \leq i \leq L$ which satisfy $[\phi_i, \phi_j] = [\phi_i, \psi_j] = [\psi_i, \psi_j] = 0$ for $1 \leq i, j \leq L$, and suppose further that all ϕ_i, ψ_i are derivations of the products $[\![\cdot,\cdot]\!]$ and $(\cdot|\cdot)$. Suppose also that $(\cdot|\cdot)$ is invariant with respect to $[\![\cdot,\cdot]\!]$. Let $\mathscr{S} \in End(U \otimes U)$ be a braiding map on U given by (2.31), and suppose $\mathscr{S}$ is locally nilpotent. Suppose also that (3.4) holds for all $a, b, c \in U$. Then there exists a logVA V containing U whose braiding map coincides with $\mathscr{S}$ on $U \otimes U$, and in which the nonnegative nth products $\mu_{(n)} := \mu_{(n+\mathscr{S})}$ restricted to $U \otimes U$ are given by*

$$\mu_{(n)}(a \otimes b) = \begin{cases} [\![a, b]\!], & n = 0, \\ (a\,|\,b)\,\mathbb{1}, & n = 1, \\ 0, & n \geq 2. \end{cases} \tag{3.20}$$

Remark 18. Observe that in the case $\mathscr{S} = 0$, the logVA V constructed in Theorem 17 reduces to the universal affine vertex algebra $V^1(\mathfrak{g})$—see Example 7—where $\mathfrak{g}$ is a Lie algebra generated by

U with Lie bracket given by $[\![\cdot,\cdot]\!]$ and bilinear form given by $(\cdot|\cdot)$. Note that in this setting, a given bilinear form $(\cdot|\cdot)$ may be replaced by any nonzero scalar multiple $\kappa(\cdot|\cdot)$ without affecting any of the required properties. It follows that we may in fact obtain $V^{\kappa}(\mathfrak{g})$ for any choice of $\kappa\in\mathbb{C}$.

Proof of Theorem 17. We divide the proof into a sequence of steps. To begin, we define the relations satisfied in the algebra $\mathcal{A}$.

First step: Defining the algebra and module.

We begin by defining an associative algebra $\mathcal{A}$ and its module V, which will serve as the logVA V. To start, let U be a vector superspace endowed with an even skew super symmetric bilinear product $[\![\cdot,\cdot]\!]: U\times U\to U$ and an even super symmetric bilinear form $(\cdot|\cdot): U\times U\to\mathbb{C}$. Let $L\geq 1$ be a positive integer, and let ϕ_i,ψ_i be nilpotent endomorphisms of U for $1\leq i\leq L$ which satisfy $[\phi_i,\phi_j]=[\phi_i,\psi_j]=[\psi_i,\psi_j]=0$ for $1\leq i,j\leq L$, and suppose further that all ϕ_i,ψ_i are derivations of the products $[\![\cdot,\cdot]\!]$ and $(\cdot|\cdot)$. Define $\mathcal{U}$ to be the vector superspace with basis

$$\mathcal{B}=\{a_n \mid a\in U,\ n\in\mathbb{Z}\}\cup\left\{\phi_i,\psi_i \mid 1\leq i\leq L\right\},$$

and let $\mathcal{T}=\bigoplus_{j\geq 0}\mathcal{U}^{\otimes j}$ be the tensor algebra over $\mathcal{U}$ with unit denoted simply by 1. We introduce a grading on $\mathcal{T}$ by setting

$$\deg a_n=-n,\quad \deg\phi_i=\deg\psi_i=\deg 1=0. \tag{3.21}$$

Define $\mathcal{A}$ to be the unital associative algebra (necessarily a quotient of $\mathcal{T}$) in which the following set of relations in $\mathcal{T}$ holds for $m,k\in\mathbb{Z}$ and $a,b\in U$:

$$\phi_\ell\otimes a_m-(-1)^{p(\phi_\ell)p(a)}a_m\otimes\phi_\ell=\left(\phi_\ell a\right)_m,\quad 1\leq\ell\leq L, \tag{3.22a}$$

$$\psi_\ell\otimes a_m-(-1)^{p(\psi_\ell)p(a)}a_m\otimes\psi_\ell=\left(\psi_\ell a\right)_m,\quad 1\leq\ell\leq L, \tag{3.22b}$$

$$a_m\otimes b_k-(-1)^{p(a)p(b)}b_k\otimes a_m=$$

$$[\![a,b]\!]_{m+k}+\delta_{m,-k}\left(m\,(a\,|\,b\,)\,1+\sum_{\ell=1}^{L}(-1)^{p(\psi_\ell)\left(p(a)+p(b)\right)}\left(\phi_\ell a\,\middle|\,b\,\right)\psi_\ell\right)$$
$$-\sum_{j\geq 1}\sum_{n=1}^{j}\sum_{\ell\in[1,L]^n}\tfrac{(-1)^j h_{0,j}(j-n)}{j!}(-1)^{\sum_{i=1}^n p(\psi_{\ell_i})p\left(\phi_{\ell_0}^{\ell_{i-1}}a\right)}\left(\phi_{\ell_1}^{\ell_n}a\right)_{m-j}\otimes\left(\psi_{\ell_1}^{\ell_n}b\right)_{k+j} \tag{3.22c}$$
$$+(-1)^{p(a)p(b)}\sum_{j\geq 1}\sum_{n=1}^{j}\sum_{\ell\in[1,L]^n}\tfrac{(-1)^j h_{0,j}(j-n)}{j!}(-1)^{\sum_{i=1}^n p(\psi_{\ell_i})p\left(\phi_{\ell_0}^{\ell_{i-1}}b\right)}\left(\phi_{\ell_1}^{\ell_n}b\right)_{k-j}\otimes\left(\psi_{\ell_1}^{\ell_n}a\right)_{m+j}.$$

The relations in (3.22) are restatements of the desired properties that should obtain in $\mathcal{A}$ and the eventual logVA V, namely, that the endomorphisms ϕ_i and ψ_i all be derivations of the nth products in V ((3.22a), (3.22b)), and that the relations in terms of the given products $[\![\cdot,\cdot]\!]$ and $(\cdot|\cdot)$ arising from the Borcherds identity at $n=0$ (2.49b) hold ((3.22c)). We denote the multiplication in $\mathcal{A}$ by juxtaposition. The grading on $\mathcal{T}$ descends in the obvious way to a grading on $\mathcal{A}$. Before proceeding, let us introduce some notation to make the relations in (3.22) less cumbersome. Let $f(j,m,k)$ be any expression in the symbols j,m,k, and define the function $F\left(j_0;a,b;m,k;f(j,m,k)\right)$, where $a,b\in U$, and $m,k,j,j_0\in\mathbb{Z}$, $j_0\geq 1$, by

$$F\left(j_0;a,b;m,k;f(j,m,k)\right):=$$
$$\sum_{j\geq j_0}\sum_{n=1}^{j}\sum_{\ell\in[1,L]^n}\tfrac{(-1)^j h_{0,j}(j-n)}{j!}(-1)^{\sum_{i=1}^n p(\psi_{\ell_i})p\left(\phi_{\ell_0}^{\ell_{i-1}}a\right)}f(j,m,k)\left(\phi_{\ell_1}^{\ell_n}a\right)_{m-j}\left(\psi_{\ell_1}^{\ell_n}b\right)_{k+j}. \tag{3.23}$$

Define $G(a,b;m,k)=F(1;a,b;m,k;1)-(-1)^{p(a)p(b)}F(1;b,a;k,m;1)$, since it will frequently occur. We may then write the relations in $\mathcal{A}$ as follows, where $[\cdot,\cdot]$ is the super commutator:

$$\left[\phi_\ell,a_m\right]=\left(\phi_\ell a\right)_m,\quad 1\leq\ell\leq L, \tag{3.24a}$$
$$\left[\psi_\ell,a_m\right]=\left(\psi_\ell a\right)_m,\quad 1\leq\ell\leq L, \tag{3.24b}$$
$$[a_m,b_k]= \tag{3.24c}$$
$$[\![a,b]\!]_{m+k}+\delta_{m,-k}\left(m\,(a\,|\,b\,)\,1+\sum_{\ell=1}^{L}(-1)^{p(\psi_\ell)\left(p(a)+p(b)\right)}\left(\phi_\ell a\,\middle|\,b\,\right)\psi_\ell\right)-G\,(a,b;m,k).$$

Observe that the set of relations (3.22) above contains infinite sums which require interpretation. We use the grading on $\mathcal{A}$ to define a topology on $\mathcal{A}$ by declaring that for any sequence $a_j b_j$, $j \to \infty$, with $\deg a_j = n - j$ and $\deg b_j = j$ for some fixed $n \in \mathbb{Z}$, we have

$$\lim_{j \to \infty} a_j b_j = 0. \tag{3.25}$$

This definition will suffice to define a topology on $\mathcal{A}$ since the summands in the infinite sums of the relations (3.22) only contain infinite sums whose terms are of the form specified above. It is clear that the algebra $\mathcal{A}$ is a unital associative algebra, and we will continue to denote its unit by 1.

With the algebra $\mathcal{A}$ in place, we proceed to define its module V as follows. Let $\mathbb{1} \in V$ be such that

$$a_n \mathbb{1} = \phi_i \mathbb{1} = \psi_i \mathbb{1} = 0, \qquad n \geq 0, \quad 1 \leq i \leq L, \tag{3.26}$$

and let V be generated as an $\mathcal{A}$-module by (3.26) and the relations obtaining in $\mathcal{A}$ from (3.24). We also identify the elements of U via $u := u_{-1}\mathbb{1}$. The following lemma characterizes the elements of V.

Lemma 19. *The superspace V defined above is the linear span of the set*

$$\left\{ \mathbb{1}, a^1_{n_1} \cdots a^r_{n_r} \mathbb{1} \mid n_i \in \mathbb{Z}_{<0}, a^i \in U, 1 \leq i \leq r \right\}. \tag{3.27}$$

Proof. The proof is almost completely the same as for Lemma 4.7 in [BV22]. We proceed by induction on the number r of generators g_i of $\mathcal{A}$ in a monomial $g_1 \cdots g_r \mathbb{1}$; we shall prove that such a monomial can be rewritten as a linear combination of monomials, each with r or fewer factors such that each generator has strictly positive degree—$\deg g_i > 0$ for all i.

The base cases $r = 0$ and $r = 1$ are obvious; assume that $r \geq 2$. Applying the inductive hypothesis, we assume $\deg g_i > 1$ for $2 \leq i \leq r$. There is nothing to prove if $\deg g_1 > 0$, so assume $\deg g_1 \leq 0$. Observe that in every relation of $\mathcal{A}$ in (3.24), the commutator brackets are equal to sums of terms that are either already in the form sought, or have a first factor whose degree is strictly greater than $\deg g_1$. Rewriting $g_1 g_2 = (-1)^{p(g_1)p(g_2)} g_2 g_1 + \left[g_1, g_2\right]$, we obtain

$$g_1 \cdots g_r \mathbb{1} = (-1)^{p(g_1)p(g_2)} g_2 g_1 g_3 \cdots g_r \mathbb{1} + \left[g_1, g_2\right] g_3 \cdots g_r \mathbb{1}. \tag{3.28}$$

In the first term of (3.28), we can again apply the inductive hypothesis to $g_1 g_3 \cdots g_r \mathbb{1}$, while in the second term, taking into account the observation just made, we may repeat the process, obtaining a finite number of terms of the claimed form. The result follows. □

Note that Lemma 19 implies that V is graded by non-negative integer degree, and $\deg \mathbb{1} = 0$. Also, it follows that $a_n v = 0$ for $n \gg 0$ and $a \in \mathcal{A}$, $v \in V$.

In order to apply the Existence Theorem, we require the fields $Y(a, z)$ corresponding to the elements $a \in U$. These are gotten by collecting the modes of each $a \in U$ into a field via (2.36). The modes of $a \in U$ are given by $a_{(n+\mathscr{S})} = a_n$, for $n \in \mathbb{Z}$.

Second step: Translation operator and translation invariance.

In order to satisfy the axioms of a logVA, we must have a translation operator T. We define T according to the properties it must satisfy, along with the initial data (the definition of $\mathscr{S}$). We take

$$T\mathbb{1} = 0, \tag{3.29a}$$

$$[T, Y(a, z)] = D_z Y(a, z), \tag{3.29b}$$

$$[T, a_n] = -(n + \mathscr{S}) a_{n-1}, = -n a_{n-1} - \sum_{\ell=1}^{L} (-1)^{p(\psi_\ell)p(a)} \big(\phi_\ell(a)\big)_{n-1} \psi_\ell, \tag{3.29c}$$

$$[T, \phi_i] = [T, \psi_i] = 0, \quad 1 \le i \le L, \tag{3.29d}$$

where $a \in V$, and $Y(a, z)$ is the field corresponding to a. Observe that (3.29b) and (3.29c) are equivalent, and we include both simply for ease of reference. Of course, the above definition must be checked for consistency with the relations of the algebra given in Section 3.3. We do this by applying ad T to each of the relations satisfied in $\mathcal{A}$, and confirm that the resulting expressions are again contained in the ideal of relations. In fact, we may simply apply Proposition 15 to conclude that T is well-defined.

The translation covariance of the fields is a direct consequence of the construction and definition of T.

Third step: Braiding map.

The braiding map is defined as in (2.31), but it is not clear that this definition in terms of the derivations ϕ_i, ψ_i gives the required properties, so we must check these properties. However, the symmetry of $\mathscr{S}$ is indeed immediate from the assumed commutation relations of the endomorphisms ϕ_i, ψ_i, for $1 \le i \le L$, so $\mathscr{S}$ is indeed well-defined.

Fourth step: Locality.

We now construct the fields in V corresponding to the generators in $\mathcal{B}$, and verify that they are local. This will follow from the imposed relation coming from (2.49b). In fact, as in Remark 3.29 in [BV22], note that by the local nilpotency of $\mathscr{S}$, then for any $a \otimes b \otimes v$, where $a, b \in U$, and $v \in V$, the coefficients of $\delta_{\mathscr{S}_{13}}(z_1, z_2)(a \otimes b \otimes v)$ span a finite-dimensional subspace, and so we can find $N \ge 0$ so that $\mu_{(N+j)} \otimes I$ (part of the right-hand term in the Borcherds identity) is zero there for all $j \ge 0$. Choosing this N, which in general depends on all of a, b, and v, then implies locality immediately.

Final step: Applying the Existence Theorem.

We have now verified that the space V has an even vector $\mathbb{1}$ and a well-defined even endomorphism T such that $T\mathbb{1} = 0$, and by construction, $\mathscr{S}$ is a braiding map whose components commute with T by construction. Setting $\hat{\mathscr{S}} = (\mathrm{ad} \otimes \mathrm{ad})(\mathscr{S})$, we obtain an $\hat{\mathscr{S}}$–local subspace $\mathcal{V}$ spanned by the translation covariant fields given by the generators in U along with the identity I. We have verified that this $\mathcal{V}$ is complete in the sense of the Existence Theorem, so the space $\overline{\mathcal{V}}$—which is the minimal $\hat{\mathscr{S}}$–local subspace closed under $(n+\hat{\mathscr{S}})$-products—is indeed the unique $\hat{\mathscr{S}}$–local subspace of translation covariant fields closed under these products. It follows from the Existence Theorem that each $v \in V$ corresponds uniquely to a logarithmic field $Y(v,z) \in \hat{\mathcal{V}}$ such that $Y(v,z)\mathbb{1}\,|_{z=0} = v$. Then $(V, \mathbb{1}, T, Y, \hat{\mathscr{S}})$ is a logVA, as claimed. Finally, the products $\mu_{(n)}$ for $n \geq 0$ follow immediately from the relations in $\mathcal{A}$, as claimed. □

Remark 20. While the representation theory of logVAs in general has not yet been established, it should be noted that one may construct modules over the logVAs constructed here as in [FZ92]; one takes $\mathcal{A}$ in place of $\hat{\mathfrak{g}}$. Then, under certain conditions, a module over $\mathcal{A}$ is equivalent to a module over the logVA V constructed here.

3.4 PBW Theorem

After constructing the logVA V in Theorem 17, it will be useful to verify the existence of, and to have an explicit construction for, a PBW basis for V.

Our proof is closely modeled on the one found in [BM] as well as the one given in [Jac79]. In general, $\mathcal{A}$ will be the algebra defined in Section 3.3, in other words, it will be the unital associative algebra with generating set

$$\mathcal{B} = \{a_n \mid a \in B,\ n \in \mathbb{Z}\} \cup \left\{\phi_i, \psi_i \mid 1 \leq i \leq L\right\},$$

where U is a vector superspace with basis B, and we denote by $\mathcal{U}$ the superspace with basis $\mathcal{B}$, and the generators in $\mathcal{B}$ are subject to the relations (3.24). Recall further from Section 3.3 that $\mathcal{A}$ is equivalently a quotient of the tensor algebra $\mathcal{T}$ over the superspace $\mathcal{U}$ by the two-sided ideal of defining relations $\mathcal{I}$ spanned by the relations in (3.22).

In order to simplify the discussion, we will assume from now on that U is a finite-dimensional *purely even* superspace (in other words, a vector space) of dimension $N = \dim U$, and that the nilpotent endomorphisms ϕ_i, ψ_i from the previous sections of this chapter are also even.

The case of a countably-infinite superspace with no assumptions on the parity of the basis vectors for U can be handled by methods very similar to those produced here, with a few extra assumptions on the endomorphisms ϕ_i, ψ_i. This general case will perhaps be addressed in a forthcoming work; it will not differ from the current one in its main ideas.

In order to prove the existence of a PBW-basis for V, we will need to order the generators of $\mathcal{A}$. Recall that in the setting of Proposition 16 and Theorem 17, the endomorphisms ϕ_i, ψ_i are assumed to be nilpotent on U. Since U is finite-dimensional, we may apply Engel's Theorem [BM; Hum12; Jac79] to obtain a basis B for U in which all of the ϕ_i and ψ_i are simultaneously strictly upper-triangulizable. Let $B = \{u^1, \dots u^N\}$ be ordered such that the flag $U_i = \operatorname{span}(\{u^1, \dots, u^i\})$, $i = 1, \dots, N$ is invariant under all the ϕ_i, ψ_i. Explicitly, this means that $\phi_i u^j, \psi_i u^j \in U_{j-1}$ for every $1 \le i \le L$ and $1 \le j \le N$. Next, we introduce an ordering $\preceq$ on the generators of $\mathcal{A}$ as follows. We let $\phi_i \preceq \phi_j$ and $\psi_i \preceq \psi_j$ for $i \le j$, and we take $\phi_i \preceq \psi_j$ for all $1 \le i, j \le L$. We order the modes of the u^i lexicographically; that is, we declare $u^i_m \preceq u^i_n$ for $m \le n$, and $u^i_m \preceq u^j_n$ whenever $i \le j$. Finally, we let $\psi_i \preceq u^j_m$ for all i, j, and m. Then $\preceq$ is a total order on the generators of $\mathcal{A}$. We denote by $\prec$ the strict total order associated with $\preceq$.

For uniformity of symbols, let us rename the generators of $\mathcal{A}$. First, let J be an index set that is totally ordered by $\preceq$. In other words, $g_i \preceq g_j \Leftrightarrow i \preceq j$ for $i, j \in J$. We let

$$\mathcal{G} = \{g_j \mid j \in J\} \tag{3.30}$$

be the renamed set of generators of $\mathcal{A}$. Note that J has a minimal element, which we denote by $j_{\min}$. Then, let $J = J_+ \sqcup J_-$, where

$$J_- = \{j \in J \mid g_j = u^i_m \text{ for some } i \text{ and } m < 0\}. \tag{3.31}$$

In other words, the generators g_j for $j \in J_-$ are those which occur as generators of V as an $\mathcal{A}$-module. Let also $g_\emptyset = 1$, the empty product. Corresponding to this partition of the index set, let $\mathcal{G} = \mathcal{G}_+ \sqcup \mathcal{G}_-$, where

$$\mathcal{G}_- = \left\{ g_j \mid j \in J_- \right\}. \tag{3.32}$$

The following notation will be useful for expressing monomials. Let the set J^m be the set of ordered m-tuples (multi-indices) of elements in J, with $J^0 = \emptyset$. We shall also need to define the *length* $\ell(M)$ of a multi-index $M \in J^m$ to be $\ell(M) = m$, with $\ell(\emptyset) = 0$. Let us define the subset $J^m_{\preceq}$ to be the set of *ordered* multi-indices,

$$J^m_{\preceq} = \left\{ M = (j_1, \dots, j_m) \in J^m \mid j_1 \preceq \cdots \preceq j_m \right\} \tag{3.33}$$

for $m \geq 1$, and $J^0_{\preceq} = \{\emptyset\}$. Finally, let $J_{\preceq} = \bigcup_{m \geq 0} J^m_{\preceq}$.

Corresponding to the multi-index $M = (j_1, \cdots, j_m) \in J^m$, define the monomial

$$g_M = g_{j_1} \cdots g_{j_m}. \tag{3.34}$$

Recall that by Lemma 19, the monomials $\mathcal{M} := \left\{ g_{j_1} \cdots g_{j_r} \mathbb{1} \mid r \geq 0,\ g_{j_i} \in \mathcal{G}_- \right\}$ span V. We wish to show that the set of ordered such monomials form a basis for V. In this setting, we have the following version of the PBW Theorem for our logVA V.

Theorem 21 (PBW Theorem)**.** Let $\mathcal{A}$ be the algebra defined in Section 3.3 with the ordered set of generators $\mathcal{G}$ as above, and let V be the corresponding logVA constructed as in Theorem 17. We assume that $\mathcal{A}$ is purely even as in the preceding discussion. Then the set of ordered monomials

$$\mathcal{O} = \left\{ g_M \mathbb{1} \mid M = (j_1, \dots, j_m) \in (J_-)^m_{\preceq},\ m \geq 0,\ g_{j_s} \in \mathcal{G}_-,\ 1 \leq s \leq m \right\}$$

is a basis for V, where the empty product ($m = 0$) in $\mathcal{A}$ corresponds to 1.

Proof. First, let us introduce some notation and definitions. Let $\mathcal{I}$ be the ideal of relations defining $\mathcal{A}$, and let the right-hand side of the commutator of two generators $g_i, g_j \in \mathcal{A}$ be given by $C(g_i, g_j)$, $i, j \in J$. Then the relations take the form

$$[g_i, g_j] = C(g_i, g_j), \tag{3.35}$$

and the two-sided ideal of relations takes the form

$$\mathcal{I} = \operatorname{span}\left\{ t\left(g_i g_j - g_j g_i - C(g_i, g_j)\right) u \mid t, u \in \mathcal{A},\ g_i, g_j \in \mathcal{G} \right\}. \tag{3.36}$$

Explicitly, the relations in $\mathcal{I}$—in the original indexing—are for us (the even case) the following:

$$\phi_\ell u^i_m - u^i_m \phi_\ell = \left(\phi_\ell u^i\right)_m, \quad 1 \le \ell \le L, \tag{3.37a}$$

$$\psi_\ell u^i_m - u^i_m \psi_\ell = \left(\psi_\ell u^i\right)_m, \quad 1 \le \ell \le L, \tag{3.37b}$$

$$\begin{aligned} u^i_m u^j_k - u^j_k u^i_m = \\ [\![u^i, u^j]\!]_{m+k} + \delta_{m,-k}\left(m\left(u^i \,\middle|\, u^j \right) 1 + \sum_{\ell=1}^{L} \left(\phi_\ell u^i \,\middle|\, u^j \right) \psi_\ell \right) \\ - \sum_{j\ge 1} \sum_{n=1}^{j} \sum_{\ell \in [1,L]^n} \tfrac{(-1)^j h_{0,j}(j-n)}{j!} \left(\left(\phi^{\ell_n}_{\ell_1} u^i\right)_{m-j} \left(\psi^{\ell_n}_{\ell_1} u^j\right)_{k+j} - \left(\phi^{\ell_n}_{\ell_1} u^j\right)_{k-j} \left(\psi^{\ell_n}_{\ell_1} u^i\right)_{m+j} \right). \end{aligned} \tag{3.37c}$$

Recall the notation $G(a, b; m, k)$ introduced in (3.24c). Then (3.37c) takes the notationally simpler form

$$\begin{aligned} u^i_m u^j_k - u^j_k u^i_m = \\ [\![u^i, u^j]\!]_{m+k} + \delta_{m,-k}\left(m\left(u^i \,\middle|\, u^j \right) 1 + \sum_{\ell=1}^{L} \left(\phi_\ell u^i \,\middle|\, u^j \right) \psi_\ell \right) - G(u^i, u^j; m, k). \end{aligned} \tag{3.38}$$

We begin by proving that the set $\mathcal{O}$ is a spanning set. Indeed, by the Existence Theorem, the set $\mathcal{M}$ of monomials certainly spans V; it remains to see that we may re-order these monomials in every case to obtain a (sum of) members of $\mathcal{O}$. This may be done inductively, by exploiting the relations in $\mathcal{I}$. The key observation is that while the relations in $\mathcal{I}$ are quadratic, the terms on the right-hand side of each relation is either of smaller length, or lower in the ordering established above. It follows that in any (unordered) monomial $g_{i_1}g_{i_2}\cdots g_{i_r}\mathbb{1}$, we may assume by induction that monomials of length less than fixed r are ordered, and that for fixed length r or less, any monomial with leading term lower in the order is also ordered. The base cases are of course trivial, since they involve one or no generator. Applying the relation in view of the comments above, we may replace, if needed, $g_{i_1}g_{i_2}$ by $g_{i_2}g_{i_1}+C(g_{i_1},g_{i_2})$. Then by induction, the term $g_{i_1}g_{i_3}\cdots$ is already ordered or, since it is of smaller length, may be ordered by induction.

The more serious claim is the linear independence of $\mathcal{O}$. We prove the linear independence of the set $\mathcal{O}$ by defining the action of $\mathcal{A}$ on an abstract vector space W with basis $\mathcal{B}_W=\left\{z_M \mid M\in J_{\preceq}\right\}$ (this portion of the proof in particular closely follows the one in [BM]). Let us define a concatenation operation in $J_{\preceq}$ as follows. If $j\in J$ and $M=(j_1,\dots,j_m)\in J_{\preceq}$, we let

$$jM=(j,j_1,\dots,j_m) \tag{3.39}$$

whenever $j\preceq j_1$, and we also let $j\preceq\emptyset$ and $j\emptyset=(j)$ for all $j\in J$. Then the claim of linear independence will follow from the following lemma.

Lemma 22. *W can be made into an $\mathcal{A}$-module with the action given by $g_jz_M=z_{jM}$ for $j\in J$, $M\in J_{\preceq}$ and $j\preceq M$.*

Proof. The proof of the lemma will be in two steps. In the first step, we define the action inductively, and in the second, we verify that it is well-defined. It will suffice to define the action of $\mathcal{A}$ on the basis $\mathcal{B}_W$ of W by the generators of $\mathcal{A}$, and extend this to an action of all of $\mathcal{A}$ on all of W by imposing associativity and linearity, respectively.

We attempt to define the action of $\mathcal{A}$ on $z_M\in W$ such that the following properties hold:

(1) $g_jz_M=z_{jM}$ whenever $j\preceq M$,

(2) $g_jz_M\in\operatorname{span}\left\{z_N \mid N\in J_{\preceq},\ \ell(N)\le\ell(M)+1\right\}$ for all $j\in J,\ M\in J_{\preceq}$,

(3) $g_i(g_jz_M)-g_j(g_iz_M)=C(g_i,g_j)z_M$ for $i,j\in J,\ M\in J_{\preceq}$.

If conditions (1) and (3) hold as claimed, then this will indeed endow W with the structure of an $\mathcal{A}$-module. The condition (2) is needed for the induction to proceed, and will ultimately define a filtration on W.

The induction proceeds first on the length $\ell(M)$, and for each fixed length, on the index $j \in J$. To start the induction, $\ell(M) = 0$ for $M = \emptyset$, and since by definition $j \preceq \emptyset$, we define $g_j z_\emptyset = z_{(j)}$. The second induction begins with $j = j_{\min}$, for which we have $j_{\min} \preceq M$ for every M, whence we define $x_{j_{\min}} z_M = z_{j_{\min} M}$. It is clear that in these cases, (1) and (2) both obtain.

To define the action for other elements, let $m \in J$ and $N \in J_{\preceq}$. Assume that $g_j z_M$ is already defined by induction for M with $\ell(M) < \ell(N)$ and all j, as well as for all $j \prec m$ and M with $\ell(M) = \ell(N)$, such that both (1), (2), and (3) hold for all terms defined. We proceed to define the action of g_m as follows:

$$g_m z_N = \begin{cases} z_{mN}, & \text{if} \quad m \preceq N, \\ g_n(g_m z_M) + C(g_m, g_n) z_M, & \text{if} \quad N = nM,\ n \prec m. \end{cases} \tag{3.40}$$

Note that this definition is consistent with the induction, since the terms on the right of the second definition are already inductively defined; the term $g_m z_M$ is defined since $\ell(M) < \ell(N)$, and by (2), $g_n(g_m z_M)$ must also be defined. The term $C(g_m, g_n) z_M$ is again defined since $\ell(M) < \ell(N)$. It is clear that the properties (1) and (2) obtain for the above definition of $g_m z_N$ as well.

By induction, we have defined the action for all elements of $\mathcal{A}$. It still remains, however, to check that this definition gives a *well-defined* action. In other words, we must verify condition (3); that is to say, we must verify that the relations $\mathcal{I}$ imposed on $\mathcal{A}$ act as zero on the module W with the given definition of the action.

Observe first that when $j \preceq M$, then (3) is equal to the right-hand side of (3.40) in the second case, so it suffices to take the case $M = kL$ for $k \prec j$. Now let us pause to look more closely at the relations in $\mathcal{I}$. First, observe that in the relations (3.37a) and (3.37b), since ϕ_ℓ and ψ_ℓ

are lower in the order than all the u^i_m, the just-mentioned considerations dispose of the need to verify any further in these cases. Hence, we may turn our attention to the relation (3.37c) (or equivalently, (3.38)). In this relation, we may use the symmetry of the bilinear form $(\cdot|\cdot)$, the skew-symmetry of the bracket $[\![\cdot,\cdot]\!]$, the fact that ϕ_ℓ, ψ_ℓ are derivations of both, and the obvious skew-symmetry of $G(a,b;m,k)$ under the swap $u^i_m \longleftrightarrow u^j_n$ to conclude that the relation $C(u^i_m, u^j_n)$ is skew-symmetric. Explicitly, we have

$$\begin{aligned}
&u^j_k u^i_m - u^i_m u^j_k \\
&= [\![u^j, u^i]\!]_{m+k} + \delta_{k,-m}\left(k\left(u^j \,\middle|\, u^i\right)1 + \sum_{\ell=1}^{L}\left(\phi_\ell u^j \,\middle|\, u^i\right)\psi_\ell\right) - G(u^j, u^i; k, m) \qquad (3.41)\\
&= -[\![u^i, u^j]\!]_{m+k} + \delta_{m,-k}\left(-m\left(u^i \,\middle|\, u^j\right)1 + \sum_{\ell=1}^{L} -\left(u^i \,\middle|\, \phi_\ell u^j\right)\psi_\ell\right) + G(u^i, u^j; m, k)\\
&= -\left(u^i_m u^j_k - u^j_k u^i_m\right).
\end{aligned}$$

It follows that we may assume $j \prec i$. Assuming this, and making the substitution $z_M = g_k z_L$, then (3) will be a consequence of

$$g_i(g_j(g_k z_L)) - g_j(g_i(g_k z_L)) - C(g_i, g_j)(g_k z_L) = 0, \tag{3.42}$$

where $k \prec j \prec i$ and $k \preceq L$. Note also that if $i \preceq k$, then $i \preceq M$, so that by induction, the above (3.42) obtains. Then the well-definedness of the action hinges on verifying (3.42). This follows by a straightforward but long and unenlightening calculation, which we suppress. The computation relies upon the application of the properties used in (3.41) as well as on the modified Jacobi identity (3.4), but is otherwise unremarkable.

□

Having shown that there exists a well-defined action of $\mathcal{A}$ on the abstract module W, the linear independence of $\mathcal{O}$ follows immediately. Indeed, given a relation $\sum_{M \in J_\preceq} \alpha_M g_M = 0$ for scalars α_M, applying it to $z_\emptyset \in W$ gives $\sum_{M \in J_\preceq} \alpha_M z_M = 0$ by the definition of the action, whence by the linear independence of the z_M, we have that $\alpha_M = 0$ for all M. We conclude that $\mathcal{O}$ is linearly independent. □

Note that Theorem 21 guarantees that the logVA V shown to exist in Theorem 17 is not trivial, which was not apparent a priori.

3.5 Construction of non-local PVAs

We apply [BV23, Theorem 3.3] to the logVAs constructed in Theorem 17 to obtain new examples of non-local PVAs. We also characterize these as symmetric algebras—more specifically, they will be differential algebras in the set of generators of the logVAs.

Compiling the aforementioned theorems, we obtain the following consequence, stated here as a theorem for emphasis:

Theorem 23 (Non-local PVA)**.** *Let U be a vector superspace endowed with an even skew super symmetric bilinear product $[\![\cdot,\cdot]\!] : U \times U \to U$ and an even super symmetric bilinear product $(\cdot|\cdot) : U \times U \to \mathbb{C}$, subject to the conditions in Theorem 17. Then the symmetric algebra $\mathcal{V} = S(\mathbb{C}[\partial] \otimes U)$ has the structure of a non-local PVA, and may be realized as a differential algebra in the set of generators U. The λ-bracket in $\mathcal{V}$ is given explicitly for generators $a, b \in U$ by*

$$\{a_\lambda b\} = [\![a, b]\!] + \lambda(a\,|\,b) + \mu_{(-1)}\left(\mathscr{S}\left(\frac{1}{\lambda+\partial} a \otimes b\right)\right), \tag{3.43}$$

where the product $\mu_{(-1)}$ is the product in the (symmetric) differential algebra $\mathcal{V}$, and is extended by sesquilinearity and the Leibniz rule.

Proof. The existence of the non-local PVA structure on $\mathcal{V}$ is a direct consequence of [BV23, Theorem 3.3] and the Existence Theorem (Theorem 17), while the realization as a symmetric algebra is a direct consequence of the PBW Theorem (Theorem 21). The formula for the λ-bracket follows from the nonnegative nth products on U and (2.58). □

CHAPTER

4

EXAMPLES OF LOGARITHMIC VERTEX ALGEBRAS

In this chapter, we present several examples of spaces U equipped with products $[\![\cdot,\cdot]\!]$ and $(\cdot|\cdot)$ satisfying the properties in Theorem 17, from which logVAs and subsequently, non-local PVAs, can be obtained from the general constructions in Chapter 3. These are fundamentally new examples of logVAs and non-local PVAs, and overcome to a fair extent some of the challenge of constructing examples of such algebras.

4.1 Examples of logVAs arising from Lie superalgebras

Let $\mathfrak{g}$ be a Lie superalgebra with Lie (super) bracket $[\cdot,\cdot] : \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$, and let $(\cdot|\cdot) : \mathfrak{g}\times\mathfrak{g}\to\mathbb{C}$ be a nondegenerate super symmetric invariant (with respect to $[\cdot,\cdot]$) bilinear form on $\mathfrak{g}$. Let also ϕ_i, ψ_i for $1 \le i \le L$ and $L \ge 1$ be mutually commuting derivations of $\mathfrak{g}$ which are also derivations of the two products $[\cdot,\cdot]$ and $(\cdot|\cdot)$. If the modified Jacobi identity (3.4) holds for all $a, b, c \in \mathfrak{g}$, then defining the non-negative nth products as in Theorem 17, we obtain a logVA $V_{\mathfrak{g}}$ generated by $\mathfrak{g}$, as in Theorem 17, if we take $[\![\cdot,\cdot]\!] = [\cdot,\cdot]$, and let the bilinear form on $\mathfrak{g}$ coincide with the one from Theorem 17. It follows that many examples of logVAs may be obtained in this way as long as the aforementioned conditions are satisfied. It should be emphasized that these are novel examples of logVAs.

Example 9. As a first example, let $\mathfrak{g} = \mathfrak{sl}(2)$, which, as is well-known, is a semisimple Lie algebra (purely even) of dimension 3, and take $(\cdot|\cdot)$ to be a scalar multiple of the Killing form, which is symmetric, invariant, and bilinear, and which, by Cartan's criterion, is nondegenerate. Let $L = 1$, and take $\phi_1 = \psi_1 = \phi$ to be a derivation of $\mathfrak{g}$ which is invariant with respect to the bracket and bilinear form. It is well-known [BM; Hum12] that such derivations exist and must be inner.

Observe that for a, b, c in a basis for $\mathfrak{g}$, the left-hand side of (3.4) must equal zero by the Jacobi identity for Lie algebras. A straightforward calculation also shows that the right-hand side of (3.4) is orthogonal (with respect to $(\cdot|\cdot)$) to each of a, b, c, whereby the nondegeneracy of the form implies that the right-hand side must also be identically zero. We conclude that the conditions of Theorem 17 are satisfied in this special case, and we obtain a logVA containing $\mathfrak{g}$ as an embedded generating subspace.

Example 10. A more interesting example is furnished by taking $\mathfrak{g} = \mathfrak{gl}(1|1)$ (see Chapter 2, Example 6 for the Lie superbrackets in this Lie superalgebra). It is known [CMY21] that $\mathfrak{g}$ is equipped with a nondegenerate even invariant supersymmetric bilinear form $(\cdot|\cdot)$ such that

$$(N\,|\,E) = (E\,|\,N) = 1, \quad \left(\psi^+ \,\middle|\, \psi^-\right) = -\left(\psi^- \,\middle|\, \psi^+\right) = 1, \tag{4.1}$$

and zero elsewhere. Take the bilinear form in Theorem 17 to be any scalar multiple of this form. Define the even endomorphism δ on the basis elements of $\mathfrak{g}$ by

$$\delta N = -\psi^+ + \psi^-, \quad \delta E = 0, \quad \delta\psi^\pm = E. \tag{4.2}$$

and extend it linearly to all of $\mathfrak{g}$. Note that this derivation is inner; in fact, $\delta = \mathrm{ad}(\psi^+ + \psi^-)$. Consequently, it follows that δ is a derivation of both the Lie superbracket and our scalar multiple of the bilinear form given above in (4.1). Furthermore, direct computation involving simple case study shows that the modified Jacobi identity (3.4) is satisfied with $L = 1$ and $\phi_1 = \psi_1 = \delta$, and $[\![\cdot,\cdot]\!]$ given by the Lie superbracket on $\mathfrak{g}$ in that setting. It follows that we may apply Theorem 17 to obtain a logVA based on $\mathfrak{gl}(1|1)$.

There is much existing literature on structures related to $\mathfrak{gl}(1|1)$, see for instance [RS92]. The relationship between our constructed logVA and other structures related to $\mathfrak{gl}(1|1)$ has, however, not yet been thoroughly investigated.

Example 11. Consider the Nappi–Witten Lie algebra $\overline{\mathfrak{h}}_4$ with basis $\mathcal{B} = \{E, F, I, J\}$ with (nonzero) Lie brackets given by

$$[E,F] = I, \quad [J,E] = E, \quad [J,F] = -F. \tag{4.3}$$

This is a solvable Lie algebra with center spanned by the single element I. It is known (e.g. [BKRS21]) that $\overline{\mathfrak{h}}_4$ has a two-parameter family of nondegenerate symmetric invariant bilinear forms, but for our purposes, we may specialize to any scalar multiple of the form $(\cdot|\cdot)$, given by

$$(E|F) = (I|J) = 1, \tag{4.4}$$

extended by symmetry to all other basis pairs (and where pairings not given above are zero). Let $\delta = \mathrm{ad}_{E+F}$, and set $L = 1$ and $\phi_1 = \psi_1 = \delta$ in Theorem 17. Since this derivation is inner, it automatically satisfies the conditions on the derivations in that theorem, and since $\overline{\mathfrak{h}}_4$ is solvable, δ is nilpotent. A direct computation verifies that δ satisfies the modified Jacobi identity (3.4). It follows by Theorem 17 that we may obtain a logVA V based in $\overline{\mathfrak{h}}_4$. Moreover, by Theorem 23, we will also obtain a non-local PVA from this algebra.

4.2 Examples of logVAs not arising from Lie superalgebras

This section presents examples of logVAs arising from algebras endowed with a bilinear scalar product and a bilinear operation (bracket) which does not satisfy the Jacobi identity for Lie superalgebras. In these examples, which were found using a computerized search with a Mathematica code [Hat24] and then verified by hand, we again begin with a finite-dimensional superspace U (over $\mathbb{C}$) with a basis

$$\mathcal{A} = \{v_1, \ldots, v_n\}. \tag{4.5}$$

Let $\phi_\ell, \psi_\ell \in \mathrm{End}(V)$ for $1 \le \ell \le L$ be a set of mutually commuting nilpotent endomorphisms, and suppose that U is endowed with a nondegenerate supersymmetric bilinear form $(\cdot|\cdot) : U \times U \to \mathbb{C}$, and a bilinear skew supersymmetric product (bracket) $[\![\cdot,\cdot]\!] : U \times U \to U$. Then Theorem 17 guarantees that we obtain a logVA V generated by U after defining the nonnegative nth products of V by (3.20), reproduced below for convenience.

$$\mu_{(n)}(a \otimes b) = \begin{cases} [\![a, b]\!], & n = 0, \\ (a \,|\, b)\,\mathbb{1}, & n = 1, \\ 0, & n \ge 2, \end{cases} \tag{4.6}$$

as long as the Borcherds identity holds, ϕ_ℓ, ψ_ℓ are derivations of the nth products, and the braiding map in V is given by (2.31).

In the case $L = 1$, $\phi_1 = \psi_1 =: \phi$ of a single derivation, we use a result in [Bak16; Hua10] which states, among other things, that in a vector space V equipped with a nondegenerate symmetric bilinear form $(\cdot|\cdot)$ and a nilpotent linear operator ϕ satisfying

$$\big(\phi(a)\,\big|\,b\big) = -\big(a\,\big|\,\phi(b)\big), \qquad \text{for all } a, b \in V, \tag{4.7}$$

V may be written as an orthogonal direct sum of vector subspaces, each of which is of one of the following forms, depending upon the parity of the dimension of the subspace.

Let $d = \dim V$ and suppose V has a basis $\{v_1, \dots, v_d\}$. Then

$$\big(v_i\,\big|\,v_j\big) = \delta_{i+j,d+1}, \tag{4.8}$$

and if d is even, say $d = 2\ell$, then

$$\phi(v_i) = \begin{cases} v_{i+1}, & 1 \le i \le \ell - 1, \\ -v_{i+1}, & \ell + 1 \le i \le 2\ell - 1, \\ 0, & i = \ell, 2\ell, \end{cases} \tag{4.9}$$

while if d is odd, $d = 2\ell - 1$, then

$$\phi(v_i) = \begin{cases} (-1)^{i+1} v_{i+1}, & 1 \le i \le 2\ell - 2, \\ 0, & i = 2\ell - 1. \end{cases} \tag{4.10}$$

Example 12. Here we present an example of a logVA structure obtained from a bracket which does not satisfy the Jacobi identity of a Lie (super)algebra. Let $\mathcal{A} = \{v_1, \dots, v_5\}$, define a non-degenerate bilinear pairing $(\cdot|\cdot)$ as in (4.8) on $\mathcal{A}$, and extend it bilinearly to all of $V = \operatorname{span}(\mathcal{A})$. Define also an even bracket $[\![\cdot,\cdot]\!]$ on $\mathcal{A}$ by

$$\begin{aligned} &[\![v_1, v_2]\!] = i\,v_1 + t\,v_3, \quad &&[\![v_1, v_3]\!] = -i\,v_2 - t\,v_4, \quad &&[\![v_1, v_4]\!] = i\,v_3, \quad &&[\![v_1, v_5]\!] = -i\,v_4, \\ & &&[\![v_2, v_3]\!] = t\,v_5, &&[\![v_2, v_4]\!] = 0, &&[\![v_2, v_5]\!] = i\,v_5, \end{aligned}$$

$$\llbracket v_3, v_4 \rrbracket = -i\, v_5, \qquad \llbracket v_3, v_5 \rrbracket = 0, \qquad (4.11)$$

$$\llbracket v_4, v_5 \rrbracket = 0,$$

where t is a complex parameter, and $i = \sqrt{-1}$, and extend it by skew symmetry and bilinearity to V. Let $\phi : V \to V$ be a nilpotent linear operator satisfying (4.7) and defined on $\mathcal{A}$ as in (4.10), and extended linearly to V. Here, we will let $L = 1$, and $\phi_1 = \psi_1 = \phi$ in (3.4). In order to have a logVA structure on V, we must check that the bracket and inner bilinear form defined on V satisfy the relevant portions of the Borcherds identity. In this case, this amounts to checking that the bilinear form is invariant with respect to the bracket, that ϕ is a derivation of the bracket, and that the modified Jacobi identity (3.4) holds. In other words, we need to verify that

$$\left(\llbracket v_i, v_j \rrbracket \,\middle|\, v_k \right) + \left(v_j \,\middle|\, \llbracket v_i, v_k \rrbracket \right) = 0 \qquad (4.12)$$

$$\phi\, \llbracket v_i, v_j \rrbracket - \llbracket \phi\, v_i, v_j \rrbracket - \llbracket v_i, \phi\, v_j \rrbracket = 0 \qquad (4.13)$$

$$\llbracket v_i, \llbracket v_j, v_k \rrbracket \rrbracket - \llbracket v_j, \llbracket v_i, v_k \rrbracket \rrbracket - \llbracket \llbracket v_i, v_j \rrbracket, v_k \rrbracket = \left(\phi\, v_i \,\middle|\, v_j \right) \phi\, v_k - \left(\phi\, v_i \,\middle|\, v_k \right) \phi\, v_j + \left(\phi\, v_j \,\middle|\, v_k \right) \phi\, v_i \qquad (4.14)$$

for every triple of vectors $v_i, v_j, v_k \in V$.

Let us name the left-hand side of (4.12) above $F(v_i, v_j, v_k)$, the left-hand side of (4.13) $G(v_i, v_j)$, the left-hand side of (4.14) $J(v_i, v_j, v_k)$, and the right-hand side of (4.14) $K(v_i, v_j, v_k)$. Let also $f(v_i, v_j, v_k) = \left(\llbracket v_i, v_j \rrbracket \,\middle|\, v_k \right)$. Note that if $J \equiv 0$ identically over all triples, then the bracket defines a Lie algebra structure on V.

By the assumed symmetry of the bilinear form, the property (4.8) that ϕ be a derivation of the bilinear form, and the skew symmetry of the bracket, the following symmetries obtain.

$$F(v_j, v_k, v_i) = -F(v_i, v_k, v_j) - F(v_k, v_j, v_i) \qquad (4.15a)$$

$$F(v_i, v_i, v_j) = F(v_i, v_j, v_i) = f(v_i, v_j, v_i) \qquad (4.15b)$$

$$F(v_i, v_j, v_j) = -2 f(v_j, v_i, v_j) \qquad (4.15c)$$

$$F(v_i, v_i, v_i) = 0 \tag{4.15d}$$

and

$$G(v_j, v_i) = -G(v_i, v_j) \tag{4.16a}$$

$$G(v_i, v_i) = 0 \tag{4.16b}$$

and finally

$$J(v_j, v_k, v_i) = J(v_i, v_j, v_k) \quad \text{and} \quad J(v_i, v_i, v_j) = 0 \tag{4.17a}$$

$$K(v_j, v_k, v_i) = K(v_i, v_j, v_k) \quad \text{and} \quad K(v_i, v_i, v_j) = 0. \tag{4.17b}$$

Note that the symmetries in (4.17) are due in part to the fact that $\left(\phi\, v_i \,\middle|\, v_i\right) = 0$ as a consequence of ϕ satisfying (4.7). By the symmetries in (4.15), it suffices to verify that (4.12) is true for all triples of distinct vectors v_i, v_j, v_k, and that $f(v_i, v_j, v_i) = 0$ for pairs of distinct vectors v_i, v_j. By the symmetries in (4.16), it suffices to verify (4.13) for pairs of distinct vectors v_i, v_j, and by the symmetries in (4.17), it suffices to verify (4.14) for triples of distinct vectors v_i, v_j, v_k. We present the verification of these equations in a few sample computations, with the remaining ones being of an entirely analogous character. Let us check (4.12) for the triple (v_1, v_2, v_4). We have

$$\begin{aligned}
\left(\llbracket v_1, v_2 \rrbracket \,\middle|\, v_4\right) + \left(v_2 \,\middle|\, \llbracket v_1, v_4 \rrbracket\right) &= (\, i\, v_1 + t\, v_3 \,|\, v_4\,) + (\, v_2 \,|\, i\, v_3\,) \\
&= i\,(\, v_1 \,|\, v_3\,) + t\,(\, v_1 \,|\, v_4\,) + i\,(\, v_2 \,|\, v_3\,) \\
&= i\,\delta_{1+3,6} + t\,\delta_{1+4,6} + i\,\delta_{2+3,6} \\
&= 0.
\end{aligned}$$

Let us also check (4.12) for the triple (v_1, v_3, v_3). We have

$$\begin{aligned} F(v_1, v_3, v_3) &= -2f(v_3, v_1, v_3) \\ &= -2\big(-[\![v_1, v_3]\!] \,\big|\, v_3\big) \\ &= 2(-i\,v_2 - t\,v_4 \,|\, v_3) \\ &= -2i\delta_{2+3,6} - 2t\delta_{4+3,6} \\ &= 0. \end{aligned}$$

We check (4.13) for the pair (v_1, v_4). We have

$$\begin{aligned} G(v_1, v_4) &= \phi\,[\![v_1, v_4]\!] - [\![\phi\, v_1, v_4]\!] - [\![v_1, \phi\, v_4]\!] \\ &= \phi\,(i\,v_3) - [\![v_2, v_4]\!] - [\![v_1, -v_5]\!] \\ &= i\,v_4 - 0 - i\,v_4 \\ &= 0. \end{aligned}$$

We also verify (4.14) for the triple (v_1, v_2, v_4) by computing both sides separately. On the left-hand side, we have

$$\begin{aligned} J(v_1, v_2, v_4) &= [\![v_1, [\![v_2, v_4]\!]]\!] - [\![v_2, [\![v_1, v_4]\!]]\!] - [\![[\![v_1, v_2]\!], v_4]\!] \\ &= [\![v_1, 0]\!] - [\![v_2, i\,v_3]\!] - [\![i\,v_1 + t\,v_3, v_4]\!] \\ &= 0 - i(t\,v_5) - i(i\,v_3) - t(-i\,v_5) \\ &= v_3, \end{aligned}$$

while on the right-hand side, we have

$$\begin{aligned} K(v_1, v_2, v_4) &= \big(\phi\,v_1 \,\big|\, v_2\big)\phi\,v_4 - \big(\phi\,v_1 \,\big|\, v_4\big)\phi\,v_2 + \big(\phi\,v_2 \,\big|\, v_4\big)\phi\,v_1 \\ &= (v_2 \,|\, v_2)(-v_5) - (v_2 \,|\, v_4)(-v_3) + (-v_3 \,|\, v_4)\,v_2 \end{aligned}$$

$$= -\delta_{2+2,6} v_5 + \delta_{2+4,6} v_3 - \delta_{3+4,6} v_2$$
$$= v_3.$$

Similar computations serve to verify that the bracket defined in (4.11) does indeed satisfy the necessary properties, and hence by our Theorem, V does extend to a logVA structure. Note that since $J(v_1, v_2, v_4) \neq 0$, the algebra generated by $\mathcal{A}$ under the given bracket is *not* a Lie algebra.

Examples of logVAs similar to the one in Example 12 are in fact quite numerous. Taking $L = 1$ and $\phi_1 = \psi_1 = \phi$ in (3.4), where ϕ is defined as in (4.9) or (4.10), examples of logVA structures exist for $n = |\mathcal{A}| = 6, 7, 8, 9, 10$ and perhaps higher. In the cases $|\mathcal{A}| = 7, 8, 9$, and 10, it seems that V be must split by the bilinear form $(\cdot|\cdot)$ into a non-trivial direct sum of mutually orthogonal subspaces in order for solutions yielding logVAs to exist, but at present, this conjecture is based merely on computational evidence.

We present a further two examples of logVAs obtained from the setup described above, one for $|\mathcal{A}| = 6$, and another for $|\mathcal{A}| = 7$. In the latter case, the bilinear form decomposes the space V into an orthogonal direct sum.

Example 13. Let $\mathcal{A} = \{v_1, \dots, v_6\}$, and define a non-degenerate symmetric bilinear pairing $(\cdot|\cdot)$ as in (4.8) on $\mathcal{A}$, extended bilinearly to $V = \text{span}(\mathcal{A})$. Let $[\![\cdot,\cdot]\!]$ define a skew symmetric even bracket on $\mathcal{A}$ by

$$\begin{aligned}
&[\![v_1, v_2]\!] = p v_1 + q v_2 + r v_3 - \frac{pq}{s} v_4, && [\![v_1, v_3]\!] = p v_2 + q v_3 + \frac{pq}{s} v_5,\\
&[\![v_1, v_4]\!] = s v_1 + \frac{2qs - 2rt - 1}{2p} v_2 - q v_4 - r v_5, && [\![v_1, v_5]\!] = t v_1 + \frac{2rt - 2qs + 1}{2p} v_3,\\
&[\![v_1, v_6]\!] = -t v_2 - s v_3 - p v_5, && \qquad - p v_4 - q v_5,\\
&[\![v_2, v_3]\!] = p v_3 - \frac{pq}{s} v_6, && [\![v_2, v_4]\!] = t v_1 + s v_2 - p v_4 + r v_6,\\
&[\![v_2, v_5]\!] = -s v_3 + q v_6, && [\![v_2, v_6]\!] = -t v_3 + p v_6, \qquad (4.18)\\
&[\![v_3, v_4]\!] = t v_2 + p v_5 + q v_6, && [\![v_3, v_5]\!] = -t v_3 + p v_6,\\
&[\![v_3, v_6]\!] = 0,
\end{aligned}$$

$$\llbracket v_4, v_5 \rrbracket = -\frac{st}{q} v_1 + t v_4 + s v_5 + \frac{2qs - 2rt - 1}{2p} v_6, \qquad \llbracket v_4, v_6 \rrbracket = \frac{st}{q} v_2 + t v_5 + s v_6,$$
$$\llbracket v_5, v_6 \rrbracket = -\frac{st}{q} v_3 + t v_6,$$

where p, q, r, s, t are complex parameters, and extend it by bilinearity to V. Let $\phi : V \to V$ be a nilpotent linear operator satisfying (4.7) and defined on $\mathcal{A}$ as in (4.9), and extended linearly to V. Here again, we will let $L = 1$, and $\phi_1 = \psi_1 = \phi$ in (3.4). In order to verify that (4.18) does indeed produce a logVA, we must verify that the properties in (4.12), (4.13), and (4.14) hold. Since the symmetries in (4.15), (4.16), and (4.17) hold, it suffices to verify these equations in the same cases as in Example 12. Again, we present a sample computation of each equation, leaving the rest to the reader.

Let us verify (4.12) for the triple (v_1, v_3, v_4). We have

$$\begin{aligned} F(v_1, v_3, v_4) &= \left(\llbracket v_1, v_3 \rrbracket \,\middle|\, v_4 \right) + \left(v_3 \,\middle|\, \llbracket v_1, v_4 \rrbracket \right) \\ &= \left(p v_2 + q v_3 + \frac{pq}{s} v_5 \,\middle|\, v_4 \right) + \left(v_3 \,\middle|\, s v_1 + \frac{2qs - 2rt - 1}{2p} v_2 - q v_4 - r v_5 \right) \\ &= p\delta_{2+4,7} + q\delta_{3+4,7} + \frac{pq}{s}\delta_{5+4,7} + s\delta_{3+1,7} + \frac{2qs - 2rt - 1}{2p}\delta_{3+2,7} - q\delta_{3+4,7} - r\delta_{3+5,7} \\ &= q - q \\ &= 0. \end{aligned}$$

Let us also verify (4.12) for the triple (v_2, v_4, v_4). We have

$$\begin{aligned} F(v_2, v_4, v_2) &= f(v_2, v_4, v_2) \\ &= \left(\llbracket v_2, v_4 \rrbracket \,\middle|\, v_2 \right) \\ &= \left(t v_1 + s v_2 - p v_4 + r v_6 \,\middle|\, v_2 \right) \\ &= t\delta_{1+2,7} + s\delta_{2+2,7} - p\delta_{4+2,7} + r\delta_{6+2,7} \\ &= 0. \end{aligned}$$

We check (4.13) for the pair (v_2, v_5). We have

$$
\begin{aligned}
G(v_2, v_5) &= \phi\, [\![v_2, v_5]\!] - [\![\phi\, v_2, v_5]\!] - [\![v_2, \phi\, v_5]\!] \\
&= \phi\left(-s\, v_3 + q\, v_6\right) - [\![v_3, v_5]\!] - [\![v_2, -v_6]\!] \\
&= -s(0) + q(0) - \left(-t\, v_3 + p\, v_6\right) + \left(-t\, v_3 + p\, v_6\right) \\
&= 0.
\end{aligned}
$$

We also verify (4.14) for the triple (v_1, v_4, v_5) by computing both sides separately. On the left-hand side, we have

$$
\begin{aligned}
J(v_1, v_4, v_5) &= [\![v_1, [\![v_4, v_5]\!]]\!] - [\![v_4, [\![v_1, v_5]\!]]\!] - [\![[\![v_1, v_4]\!], v_5]\!] \\
&= \left[\!\!\left[v_1, -\frac{s\,t}{q}\, v_1 + t\, v_4 + s\, v_5 + \frac{2q\,s - 2r\,t - 1}{2p}\, v_6 \right]\!\!\right] - \left[\!\!\left[v_4, t\, v_1 + \frac{2r\,t - 2q\,s + 1}{2p}\, v_3 \right]\!\!\right] \\
&\quad - \left[\!\!\left[s\, v_1 + \frac{2q\,s - 2r\,t - 1}{2p}\, v_2 - q\, v_4 - r\, v_5, v_5 \right]\!\!\right] \\
&= -\frac{s\,t}{q}(0) + t\left(s\, v_1 + \frac{2q\,s - 2r\,t - 1}{2p}\, v_2 - q\, v_4 - r\, v_5 \right) + s\left(t\, v_1 + \frac{2r\,t - 2q\,s + 1}{2p}\, v_3 \right) \\
&\quad + \frac{2q\,s - 2r\,t - 1}{2p}\left(-t\, v_2 - s\, v_3 - p\, v_5\right) + t\left(s\, v_1 + \frac{2q\,s - 2r\,t - 1}{2p}\, v_2 - q\, v_4 - r\, v_5 \right) \\
&\quad + \frac{2r\,t - 2q\,s + 1}{2p}\left(t\, v_2 + p\, v_5 + q\, v_6\right) - s\left(t\, v_1 + \frac{2r\,t - 2q\,s + 1}{2p}\, v_3 \right) + r(0) \\
&\quad - \frac{2q\,s - 2r\,t - 1}{2p}\left(-s\, v_3 + q\, v_6\right) + q\left(-\frac{s\,t}{q}\, v_1 + t\, v_4 + s\, v_5 + \frac{2q\,s - 2r\,t - 1}{2p}\, v_6 \right) \\
&= v_5,
\end{aligned}
$$

while on the right-hand side, we also have

$$
\begin{aligned}
K(v_1, v_4, v_5) &= \left(\phi\, v_1 \,\middle|\, v_4 \right) \phi\, v_5 - \left(\phi\, v_1 \,\middle|\, v_5 \right) \phi\, v_4 + \left(\phi\, v_4 \,\middle|\, v_5 \right) \phi\, v_1 \\
&= (\, v_2 \,|\, v_4 \,)(-v_6) - (\, v_2 \,|\, v_5 \,)(-v_5) + (-v_5 \,|\, v_5 \,)\, v_2 \\
&= -\delta_{2+4,7}\, v_6 + \delta_{2+5,7}\, v_5 - \delta_{5+5,7}\, v_2 \\
&= v_5.
\end{aligned}
$$

Similar computations serve to verify that the bracket defined in (4.18) satisfies the necessary properties, and hence by our Theorem, V does extend to a logVA structure. Note that since $J(v_1, v_4, v_5) \neq 0$, the algebra generated by $\mathcal{A}$ under the given bracket is once again *not* a Lie algebra.

Example 14. Let $\mathcal{A} = \{v_1, \ldots, v_7\}$. We split $V = \text{span}(\mathcal{A})$ into an orthogonal direct sum $V = V_1 \oplus V_2 = \text{span}(\{v_1, v_2, v_3, v_4\}) \oplus \text{span}(\{v_5, v_6, v_7\})$, and define a non-degenerate symmetric bilinear pairing $(\cdot|\cdot)$ on V in accordance with the result of [Bak16; Hua10] as follows. The nonzero ordered pairings $\left(v_i \middle| v_j \right)$, with $i \le j$, are

$$(v_1 | v_4) = (v_2 | v_3) = 1 \quad \text{and} \quad (v_5 | v_7) = (v_6 | v_6) = 1, \tag{4.19}$$

and these are extended by bilinearity and symmetry to all of V. We define a skew symmetric even bilinear bracket $[\![\cdot,\cdot]\!]$ on $\mathcal{A}$ by

$$\begin{aligned}
&[\![v_1, v_2]\!] = p\,v_7, && [\![v_1, v_3]\!] = i\,v_6 + q\,v_7, && [\![v_1, v_4]\!] = r\,v_7,\\
&[\![v_1, v_5]\!] = -r\,v_1 - q\,v_2 - p\,v_3 + s\,v_6, && [\![v_1, v_6]\!] = -i\,v_2 - s\,v_7, && [\![v_1, v_7]\!] = 0,\\
& && [\![v_2, v_3]\!] = (-i + r)v_7, && [\![v_2, v_4]\!] = 0,\\
&[\![v_2, v_5]\!] = (i - r)v_2 + p\,v_4, && [\![v_2, v_6]\!] = 0, && [\![v_2, v_7]\!] = 0,\\
& && && [\![v_3, v_4]\!] = t\,v_7,\\
&[\![v_3, v_5]\!] = -t\,v_1 + (-i + r)v_3 + q\,v_4 + u\,v_6, && [\![v_3, v_6]\!] = i\,v_4 - u\,v_7, && [\![v_3, v_7]\!] = 0,\\
&[\![v_4, v_5]\!] = t\,v_2 + r\,v_4, && [\![v_4, v_6]\!] = 0, && [\![v_4, v_7]\!] = 0,\\
& && [\![v_5, v_6]\!] = u\,v_2 + s\,v_4, && [\![v_5, v_7]\!] = 0,\\
& && && [\![v_6, v_7]\!] = 0,
\end{aligned} \tag{4.20}$$

where p, q, r, s, t, and u are complex parameters and $i = \sqrt{-1}$, and extend it by bilinearity and skew symmetry to V. Again, we verify the equations (4.12), (4.13), and (4.14) as in the previous two examples. Let us verify (4.12) for the triple (v_1, v_5, v_6). We have

$$\begin{aligned}
F(v_1, v_5, v_6) &= \left([\![v_1, v_5]\!] \,\middle|\, v_6 \right) + \left(v_5 \,\middle|\, [\![v_1, v_6]\!] \right) \\
&= \left(-r\,v_1 - q\,v_2 - p\,v_3 + s\,v_6 \,\middle|\, v_6 \right) + (v_5 | -i\,v_2 - s\,v_7) \\
&= -r(0) - q(0) - p(0) + s(1) - i(0) - s(1) \\
&= s - s \\
&= 0.
\end{aligned}$$

Let us also verify (4.12) for the triple (v_3, v_5, v_5). We have

$$\begin{aligned}
F(v_3, v_5, v_5) &= -2f(v_5, v_3, v_5) \\
&= -2\left([\![v_5, v_3]\!] \,\middle|\, v_5 \right) \\
&= 2\left(-t\,v_1 + (-i + r)v_3 + q\,v_4 + u\,v_6 \,\middle|\, v_5 \right) \\
&= -2(0) + 2(-i + r)(0) + 2q(0) + 2u(0) \\
&= 0.
\end{aligned}$$

We check (4.13) for the pair (v_1, v_5). We have

$$\begin{aligned}
G(v_1, v_5) &= \phi\, [\![v_1, v_5]\!] - [\![\phi\, v_1, v_5]\!] - [\![v_1, \phi\, v_5]\!] \\
&= \phi \left(-r\,v_1 - q\,v_2 - p\,v_3 + s\,v_6 \right) - [\![v_2, v_5]\!] - [\![v_1, v_6]\!] \\
&= -r\,v_2 - q(0) + p\,v_4 - s\,v_7 - \big((i - r)v_2 + p\,v_4 \big) - (-i\,v_2 - s\,v_7) \\
&= 0.
\end{aligned}$$

We also verify (4.14) for the triple (v_1, v_3, v_5). On the left-hand side, we have

$$\begin{aligned}
J(v_1, v_3, v_5) &= [\![v_1, [\![v_3, v_5]\!]]\!] - [\![v_3, [\![v_1, v_5]\!]]\!] - [\![[\![v_1, v_3]\!], v_5]\!] \\
&= [\![v_1, -t\,v_1 + (-i + r)v_3 + q\,v_4 + u\,v_6]\!] - [\![v_3, -r\,v_1 - q\,v_2 - p\,v_3 + s\,v_6]\!]
\end{aligned}$$

$$
\begin{aligned}
&-[\![i v_6 + q v_7, v_5]\!] \\
=&-\frac{st}{q}(0) + t\left(s v_1 + \frac{2qs-2rt-1}{2p} v_2 - q v_4 - r v_5\right) + s\left(t v_1 + \frac{2rt-2qs+1}{2p} v_3\right) \\
&+ \frac{2qs-2rt-1}{2p}\left(-t v_2 - s v_3 - p v_5\right) + t\left(s v_1 + \frac{2qs-2rt-1}{2p} v_2 - q v_4 - r v_5\right) \\
&+ \frac{2rt-2qs+1}{2p}\left(t v_2 + p v_5 + q v_6\right) - s\left(t v_1 + \frac{2rt-2qs+1}{2p} v_3\right) + r(0) \\
&- \frac{2qs-2rt-1}{2p}\left(-s v_3 + q v_6\right) + q\left(-\frac{st}{q} v_1 + t v_4 + s v_5 + \frac{2qs-2rt-1}{2p} v_6\right) \\
=& \; v_5,
\end{aligned}
$$

while on the right-hand side, we also have

$$
\begin{aligned}
K(v_1, v_4, v_5) &= \left(\phi v_1 \,\middle|\, v_4\right)\phi v_5 - \left(\phi v_1 \,\middle|\, v_5\right)\phi v_4 + \left(\phi v_4 \,\middle|\, v_5\right)\phi v_1 \\
&= (v_2 \mid v_4)(-v_6) - (v_2 \mid v_5)(-v_5) + (-v_5 \mid v_5)\, v_2 \\
&= -\delta_{2+4,7} v_6 + \delta_{2+5,7} v_5 - \delta_{5+5,7} v_2 \\
&= v_5.
\end{aligned}
$$

Similar computations serve to verify that the bracket defined in (4.18) satisfies the necessary properties, and hence by our Theorem, V does extend to a logVA structure. Note that since $J(v_1, v_4, v_5) \neq 0$, the algebra generated by $\mathcal{A}$ under the given bracket is once again *not* a Lie algebra.

CHAPTER

5

FURTHER TOPICS OF RESEARCH

In this chapter, we briefly explore some directions in which the current research might be extended, state some questions which remain unanswered, and indicate some of our own intentions for future research.

5.1 Extending the nth products

In the setting of Theorem 17, we defined the nonnegative nth products for $n = 0$ and $n = 1$ in terms of known data for the generating subspace U. This construction mimics the nonnegative nth products in the universal affine VA, and is sufficient to furnish us with novel examples of logVAs. However, a greater generality of our construction lurks; a priori, there is no immediate obstruction to extending the nonnegative nth products to arbitary fixed positive $n \geq 1$. The question then arises as to what sorts of structures or logVAs (and by extension, non-local PVAs) might be obtained in this way, if at all.

Put in another way, we would like to know if when starting from an arbitrary λ-bracket in a non-local PVA, there is a way to construct a corresponding logVA. In other words, we would like to “reverse the steps” pursued in the present work. Even the question of how to generate a logVA from a non-local LCA (possibly with more structure added) is one that remains open [BV23].

5.2 Generalizing the PBW Theorem

Another avenue for further investigation is to see to what extent the PBW Theorem presented here may be applied or generalized in the setting of logVAs, or perhaps elsewhere. While the proof of our version of the PBW Theorem was in most regards very similar to (and based upon) the proof for $U(\mathfrak{g})$ with a $\mathfrak{g}$ a Lie algebra, there were some subtle differences, in particular with respect to indexing a basis suitably for induction. In more complicated logVAs, will such subtleties prove to be hindrances to reaching a conclusion? We hope to generalize the approach taken here and enunciate PBW Thereoms even for previously (to this work) constructed logVAs [BV22; BV23; BV24].

5.3 Connections to integrable systems

While the connection to integrable systems via the construction of non-local PVAs is mentioned in the present work, it is never explicitly established. Therefore, natural questions arise around the sorts of integrable systems associated to the non-local PVAs obtained here; cf. [DSK13].

5.4 Further examples

Finally, while Theorem 17 allows for the construction of explicit (and new) examples of logVAs, a more unified characterization of the logVAs obtained by this method is unfortunately incomplete. It would be interesting to obtain, for a given family of Lie super algebras, a complete characterization of the families of commuting nilpotent derivations ϕ_i, ψ_i satisfying the modified Jacobi identity (3.4). More modestly, a set of precise conditions on, say, semisimple Lie algebras (such as $\mathfrak{sl}(n)$ for $n > 2$) for the existence of a logVA structure based (in some sense) would be a desirable end.

The examples for $\mathfrak{gl}(1|1)$ and $\overline{\mathfrak{h}}_4$ show that some solvable Lie algebras might be good candidates for constructing examples of logVAs by our method. Nonsemisimple Lie algebras may not, however, possess a nondegenerate bilinear form as in these cases, but in the case of so-called *metric* Lie algebras the case may hold hope [FOS94].

In greater generality, a better understanding of the class of *non-Lie* examples and a (perhaps rough) classification of them is at present unknown.

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