Correspondence between stress and strain-space formulation of plasticity for anisotropic materials

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1 INTRODUCTION

A constitutive model which gives stress as a function of strain is called a strain-space model. In the case of strain-space plasticity, yield conditions and flow rule are expressed as functions of strain. The particular form which constitutive equations take in this case has a number of implications especially in view of the solution of boundary value problems. Most of the finite element codes use the displacement formulation (Noor, 1981) which requires the stiffness matrix. The stiffness matrix is directly generated in a strain-space formulation while an inversion is required when using a stress-space formulation.

Furthermore, it has been shown (Naghdi and Trapp, 1975) that the loading criterion, which is different for strain-hardening and strain-softening materials in the stress-space formulation, becomes unique in the strain-space formulation.

In spite of the notable advantages of the strain-space formulation, plasticity theory is still applied using the stress-space formulation. Most of the plasticity theories currently in use may be conveniently reformulated in strain space in order to achieve a better computational efficiency: in the following the relationships between stress-space and strain-space parameters are presented for a class of classical plasticity models. Because of their simplicity and generality these results constitute a very useful tool for engineers.

2 PREVIOUS WORK

The possibility of a strain-space formulation of plasticity was first pointed out by Drucker (1950), without, however working out the details. Other investigators have approached plasticity from a strain-space perspective. Lenskii (1960), for example, developed a model for permanently deforming media, which has since been applied to soils (Palmer et al., 1973); this model avoids the use of yield surface and is not along the line of traditional plasticity. Naghdi and Trapp (1975) showed that many of the familiar features of stress-space plasticity can be carried over to strain space. They also outlined the generality of the loading criteria in strain-space plasticity, in contrast with criteria relative to the stress-space formulation which have to be modified for elastic-perfectly plastic materials. A formulation of strain-space plasticity with multiple loading surfaces has been proposed by Yoder and Iwan (1981) who also proved the equivalence between stress and strain-space loading criteria for the case of an isotropic elastic material with plastic behavior independent of the spherical part of the stress. However, this approach does not account for plastic volumetric strain with only the deviatoric part of stress and strain involved in the plastic constitutive law. Within these conditions, along with the isotropic behavior of the material, stress space and strain space differ in scale and stress and strain increments are proportional to one another the corresponding vectors are parallel. In many cases of practical interest, though, anisotropic behavior and plastic volumetric strain are to be accounted for.

Many elasto-plastic models, like the Drucker-Prager (1952) model and other cap-type models (DiMaggio and Sandler, 1971; Sandler et al., 1976; Mizuno, 1981) present plastic dilatancy, others (Prevost, 1978) present anisotropy. These models can be converted in strain space using the results presented in this paper.

A demonstration of the equivalence between stress and strain-space formulation of plasticity was presented by the author (Barbagelata, 1986) for the case of stable materials in the sense defined by Drucker (1956).

It is worth noting that no loading criteria are available in stress space for perfect plasticity and strain-softening behavior; in these cases, strain-space considerations have been implicitly used to detect loading and unloading.

3 STRESS AND STRAIN-SPACE FORMULATION OF PLASTICITY

In this section, the main results of the classical theory of elasto-plastic materials are recollected. Attention is confined to the case of small deformations. Although experiments show that normality rule is sometimes violated for soils (Prevost, 1978) and other materials that undergo permanent volume changes, it represents a powerful tool for the approach of problems in plasticity. Thus, in the present exposition, the properties of normality and convexity are retained. Lower case letters convenction is adopted for stress-space formulation and upper case letters for strain-space formulation. In Table 1 the formal correspondence between the two formulations is shown: the stress tensor $\underset{\sim}{\sigma}$ is replaced by the strain tensor $\underset{\sim}{E}$, the plastic strain $\underset{\sim}{\varepsilon}^P$ is replaced by the relaxation stress $\underset{\sim}{\Sigma}^R$, the gradient operator in stress space $\underset{\sim}{\nabla}$ is replaced by the gradient operator in strain space, which has the same symbol.

4 CORRESPONDENCE BETWEEN MODEL PARAMETERS

Based on the demonstration of the equivalence between stress and strain-space formulations of plasticity (Barbagelata, 1986), it is possible to determine the relationships between corresponding model parameters for a given constitutive model. Such relationships are summarized in Table 2.

Table 1. Summary of stress and strain-space formulation of plasticity

Formulation	Stress space	Strain space
Yield surface	$f(\underset{\sim}{\sigma},\underset{\sim}{\varepsilon}^{P},k) = 0$	$F(\underset{\sim}{\varepsilon},\underset{\sim}{\sigma}^{R},K) = 0$
Hardening	$\overset{\circ}{k} = \underset{\sim}{d}^{T} \cdot \overset{\circ}{\underset{\sim}{\varepsilon}}{}^{P}$	$\overset{\circ}{K} = \underset{\sim}{D}^{T} \cdot \overset{\circ}{\underset{\sim}{\sigma}}{}^{R}$
Normal to the Yield Surface	$\underset{\sim}{n} = \frac{1}{g}\underset{\sim}{\nabla} f(\underset{\sim}{\sigma})$ $g = \lVert \underset{\sim}{\nabla}\ f(\underset{\sim}{\sigma}) \rVert$	$\underset{\sim}{N} = \frac{1}{G}\underset{\sim}{\nabla} F(\underset{\sim}{\varepsilon})$ $G = \lVert \underset{\sim}{\nabla}\ F(\underset{\sim}{\varepsilon}) \rVert$
Loading Criterion	$f < 0 \longrightarrow$ unloading $f = 0,\ \underset{\sim}{n}^{T}\cdot\dot{\underset{\sim}{\sigma}} < 0 \longrightarrow$ unloading $f = 0,\ \underset{\sim}{n}^{T}\cdot\dot{\underset{\sim}{\sigma}} = 0 \longrightarrow$ neutral loading $f = 0,\ \underset{\sim}{n}^{T}\cdot\dot{\underset{\sim}{\sigma}} > 0 \longrightarrow$ loading	$\longleftarrow F < 0$ $\longleftarrow F = 0,\ \underset{\sim}{N}^{T}\cdot\dot{\underset{\sim}{\varepsilon}} < 0$ $\longleftarrow F = 0,\ \underset{\sim}{N}^{T}\cdot\dot{\underset{\sim}{\varepsilon}} = 0$ $\longleftarrow F = 0,\ \underset{\sim}{N}^{T}\cdot\dot{\underset{\sim}{\varepsilon}} > 0$
Flow Rule	$\overset{\circ}{\underset{\sim}{\varepsilon}}{}^{P} = \frac{1}{h}(\underset{\sim}{n}^{T}\cdot\overset{\circ}{\underset{\sim}{\sigma}})\underset{\sim}{n}$	$\overset{\circ}{\underset{\sim}{\sigma}}{}^{R} = \frac{1}{H}(\underset{\sim}{N}^{T}\cdot\overset{\circ}{\underset{\sim}{\varepsilon}})\underset{\sim}{N}$

Table 2. Correspondence between model parameters

Yield Surface	$F(\underset{\sim}{\varepsilon},\underset{\sim}{\sigma}^{R},K) = f[\underset{\sim}{C}\underset{\sim}{\varepsilon} - \underset{\sim}{\sigma}^{R}, \underset{\sim}{C}^{-1}\underset{\sim}{\sigma}^{R}, k]$; $K = k$
Normal to the Yield Surface	$\underset{\sim}{N} = \frac{g}{G}\ \underset{\sim}{C}\cdot\underset{\sim}{n}$
Hardening Law	$H = \frac{g^{2}}{G^{2}}(h + \underset{\sim}{n}^{T}\cdot\underset{\sim}{C}\cdot\underset{\sim}{n})$; $\underset{\sim}{D} = \underset{\sim}{C}^{-1}\cdot\underset{\sim}{d}$
Plastic Flow	$\overset{\circ}{\underset{\sim}{\sigma}}{}^{R} = \underset{\sim}{C}\cdot\overset{\circ}{\underset{\sim}{\varepsilon}}{}^{P}$

Notes: $\underset{\sim}{C}$ is the 6 by 6 symmetric matrix of elastic coefficients. $\dot{\underset{\sim}{\sigma}}$ and $\dot{\underset{\sim}{\varepsilon}}$ are the six component vectors of stress rate and strain rate, respectively.

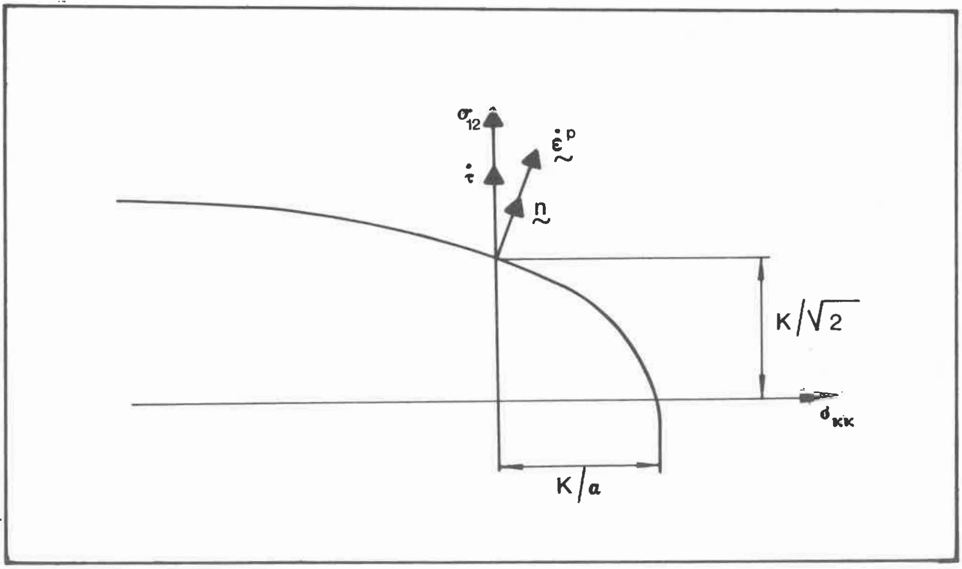

Figure 1. Yield surface in stress-plane $\sigma_{12} - \sigma_{kk}$ for a Drucker-Prager material.

5 EXAMPLE OF APPLICATION

As an example of application of the equivalence between stress and strain-space formulation a simple problem is presented where the effect of hydrostatic pressure on shearing resistance of the material is considered using Drucker-Prager (1952) model. The yield function can be written as follows (Chen, 1982):

$$f\ (\sigma_{ij},\ \varepsilon^P_{ij},\ k) = \sqrt{s_{ij}\ s_{ij}} + \alpha\sigma_{kk} - k \tag{1}$$

where s_{ij} represents a generic component of the deviatoric part of the stress tensor.

The problem illustrated in Figure 1 is considered, in which all the stresses are zero except the shear stress $\sigma_{12} = \tau$. The stress will hit the yield surface when $\tau = k/\sqrt{2}$, then being the loading criterion satisfied, plastic deformation will take place according to the flow rule:

$$\overset{\circ}{\gamma} = \overset{\circ}{\gamma}^E + \overset{\circ}{\gamma}^P = \overset{\circ}{\tau}/\mu + 2\ \overset{\circ}{\tau}/(hg^2) \tag{2}$$

$$e = e^P = \sqrt{2}\,\alpha\overset{\circ}{\tau}/(hg^2) \tag{3}$$

where e is the spherical part of the strain and μ the elastic shear modulus.

In solving the problem in strain space the relationships provided in Table 2 can be used. The following expression is obtained for the yield surface:

$$F(\varepsilon_{ij}, \varepsilon_{ij}^P, k) = 2\mu \Theta + 3\alpha \bar{K}(\varepsilon_{kk} - \varepsilon_{kk}^P) - k \tag{4}$$

where:

$$\Theta = [(\eta_{ij} - \eta_{ij}^P)(\eta_{ij} - \eta_{ij}^P)]^{\frac{1}{2}} \tag{5}$$

η_{ij} is the deviatoric part of the strain tensor and $\bar{K}$ the elastic bulk modulus. The plastic strain components are related to the relaxation stress component by the last equation in Table 2. The equation for the normal to the relaxation surface can be obtained applying the corresponding equation in Table 2:

$$N_{ij} = \frac{1}{G}\left[\frac{2\mu}{\Theta}(\eta_{ij} - \eta_{ij}^P) + 3\alpha \bar{K}\delta_{ij}\right] \tag{6}$$

where

$$G = (4\mu^2 + 27\alpha^2 \bar{K}^2)^{\frac{1}{2}} \tag{7}$$

For strain states inside the relaxation surface, the only non-zero strain is $\varepsilon_{12} = \gamma/2$; plastic flow begins when $\gamma = k/(\sqrt{2}\mu)$ as it can be seen substituting this value in equation (4). By using the expressions provided in Table 1 for the strain-space formulation the dual problem in strain space can be solved, by imposing the time history of strains given by expressions (2) and (3), to obtain the corresponding stress history.

In doing so, the original stress history used as input for the solution of the problem in stress space is obtained, that is, all stresses are found to be zero except the tangential stress σ_{12} which is actually equal to τ.

6 CONCLUSIONS

In order to achieve a better computational accuracy, it is suggested that engineers facing the problem of implementing a plastic constitutive model, consider the possibility of using a strain-space formulation. Due to the fact that most of the existing models are, in origin, formulated in stress space, a conversion of the model to strain space may be needed. The general relationships between stress and strain-space parameters in the mechanical theory of plasticity, provided in this paper, represents a useful tool for determining the model parameters in strain space.

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